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Three essays in dynamic macroeconomics

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DEDICATION

I dedicate this dissertation to my husband Zhe, and my parents, Yueqin and Yongxiang, without whose support I would not have been able to complete this work, and to my children Jingying and Jiaying, who are the source of my motivation.

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CHAPTER 1

PAY-AS-YOU-GO PENSIONS AND ENDOGENOUS RETIREMENT

1.1 Introduction

Over the time-frame of a century, the social security system has emerged as a powerful institution, one that has played a complex yet important role in the evolution of most developed countries. Currently, nearly 150 countries offer some sort of old-age pension for its citizens. In a stylized sense, the financing of the old-age pension system can occur in two different ways. First, the system can be funded, implying that the mandatory contributions of the young are invested today and returned to them with interest in the next period. Second, the system can be unfunded, implying pensions to the current retired, old are financed by contributions from the young and working generation period by period. Accordingly, the latter system is usually referred to as a pay-as-you-go (PAYG) system.

Most countries that offer a public retirement provision finance these transfers on a PAYG basis. Nevertheless, the rationale for such a system -- even whether it ought to be introduced -- continues to be hotly debated among policymakers. Alongside the policy dimensions of the debate, the consequences of PAYG programs are also a disputed subject within the realm of economic models. In particular, the rationale for such programs in the theoretical literature is not clear-cut. For instance, within a standard overlapping generations (OG) model where the young supply their labor inelastically and the old are retired, a well-established result is that a PAYG system will reduce the incentive to save and thereby reduce capital accumulation (see Diamond (1965) for the original analysis). In contrast, the funded pension system has no effect on aggregate saving and capital accumulation. In short, the funded system is neutral but the PAYG system is not. Attached to this neutrality result is a classic result regarding desirability. If long-run (or steady-state) welfare is the criterion, then it is well-known that if the economy is initially dynamically efficient -- the net interest rate is greater than the population growth rate -- then there is no welfare case for introducing a PAYG system (see Aaron (1966) and Samuelson (1975)). And as documented in Aaron (1966) and Abel et al. (1989), most developed countries are dynamically efficient. By implication, a PAYG system in these countries is not desirable and a funded system is preferred.

Despite this, most countries have implemented (or are considering implementing) pension programs with a substantial PAYG component. The literature has offered several justifications. For example, the role played by unfunded pension systems in ameliorating idiosyncratic risks in worlds with incomplete financial markets has been surveyed in Krueger (2006). Andersen and Bhattacharya (2011) and Caliendo and Gahramanov (2011) study the importance of agent myopia in providing a justification for such pension systems. Fuster et al. (2007) introduce bidirectional altruism along with mortality and earnings risks in a framework similar to one adopted by Conesa and Krueger (1999). Cooley and Soares (1999) explore a political-economy justification for PAYG pensions.

Most of these studies, however, take us far away from the world of the baseline Diamond model by introducing many different additional features. Clean insights are lost in the process. In this paper, we return to the original spirit of the Diamond (1965) model and uncover a novel channel of justifying PAYG pensions therein. This way we use the cleanest possible environment to capture all equilibrium responses. Our focus on the pension role of social security prevents the analysis from getting confounded with all other roles of social security (such as, its redistribution or insurance role).

Recall that in the standard Diamond OG model, and in much of the ensuing literature, it is assumed agents work full time when young and retire at the beginning of the second period of their life. In other words, their old-age labor supply is arbitrarily fixed at zero, implying retirement is exogenously imposed. This assumption is especially troublesome since it is readily apparent that retirement decisions are not made in a vacuum and clearly respond to the pension system itself. In this paper, we introduce an endogenous retirement decision in the second period of life, leaving other elements of the Diamond (1965) environment intact.[1] We start with a very general formulation of the environment, i.e., a time-separable utility function, a neoclassical production function, and a PAYG governmental budget constraint. Indeed, one of the unique elements of our analysis is the broad generality of the environment. Many partial and general-equilibrium results are derived within this setting. In places, however, tractability is achieved by assuming C.E.S preferences.

Within this new framework, several results regarding the effects of introducing a PAYG system on the capital-labor ratio and on long-run welfare are derived. First, in a general environment with endogenous retirement, the effect of a PAYG system on the capital-labor ratio is ambiguous -- it may be decreasing, increasing or even neutral. Second, in the case of CES utility, the effect of a PAYG system on the capital-labor ratio only depends on the elasticity of substitution between consumptions in both periods and old-age leisure. Specifically, if the elasticity of substitution is equal to one, i.e., with logarithmic utility, introducing a PAYG system is perfectly neutral in terms of the capital-labor ratio. Third, it is also shown that there may be long-run welfare

[1] There have been a few studies on the relationship between endogenous labor supply and social security. For example, Andersen and Bhattacharya (2013) allow for endogenous labor supply in the first period of life and show that, under a sufficient condition, the old agents are no less risk-averse than the young, the Aaron-Samuelson result still holds. Hu (1979) and Michel and Pestieau (1999) introduce endogenous retirement in the second period. Hu (1979) shows that with a bequest motive, the impact of an increase in retirement benefits on capital accumulation is ambiguous. Michel and Pestieau (1999) find the rate of participation declines as the size of social security system increases.

gains from introducing a PAYG pension if the initial steady-state is dynamically efficient. The results regarding the welfare effects crucially depend on how the capital-labor ratio changes.

These results serve to extend our understanding of the role played by PAYG systems. The main insight is, in a model with endogenous retirement, changes in the capital-labor ratio depend not only on how saving responds to the pension system, but also on how older workers adjust their labor supply -- their decision on when to retire -- in response to the system itself. It is this latter effect that has been ignored in the literature. With a distorting payroll tax used to finance such a system, it is possible for old workers to change their labor supply and even bring about an increase in the capital-labor ratio, which, in turn, may raise long-run welfare if the economy is initially dynamically efficient. These results are in sharp contrast to those known for many decades in the literature.

The plan for the rest of the paper is as follows. Section 1.2 outlines the environment of the model, a generalization of Diamond (1965) to endogenous retirement. In section 1.3 we study the behavior of the consumer. Section 1.4 describes the general equilibrium and the effect of the PAYG system on the capital-labor ratio. In section 1.5 we analyze the welfare effects, and section 1.6 provides concluding remarks. Proofs of all major results can be found in the appendices.

1.2 The Model

1.2.1 Preliminaries

The economy is closed and consists of firms, an infinite sequence of two period lived overlapping generations, an initial old generation, and an infinitely lived government. Let $t = 1, 2, \ldots$ index time. At each date t, a new generation composed of a continuum of identical young agents appears. Population is assumed to be stationary and the number of individuals in each generation is normalized to one.

Identical firms hire the available labor force L_t, and the aggregate capital stock K_t, to produce the aggregate output. Production occurs according to a neoclassical production function $F(K_t, L_t)$, exhibiting constant returns to scale. The homogenous output can be used both as a consumption good and as an investment good. The capital stock is assumed to depreciate completely in each period.[2] Hence, investment in period t determines the capital stock in period $t+1$, i.e., $I_t = K_{t+1}$. The initial old agents are endowed with K_1 units of capital. Let $k_t \equiv K_t / L_t$ denote the capital-labor ratio. Output per unit of labor is $f\left(k_t\right) \equiv F\left(K_t / L_t, 1\right)$ where $f\left(k_t\right)$ is the intensive production function. The standard neoclassical assumptions for the production function apply, i.e., for all $k > 0$, $f' > 0$, $f'' < 0$, and the Inada conditions hold.

Firms are assumed to maximize profits, given by $\left\{F(K_t, L_t) - w_t L_t - R_t K_t\right\}$, where w_t and R_t are respectively the competitive wage rate and the gross real return between $t-1$ and t. As markets are perfectly competitive, the input factors are rewarded by their marginal products:

$$w_t \equiv w\left(k_t\right) = f\left(k_t\right) - k_t f'\left(k_t\right), \tag{1.1}$$

$$R_t \equiv R\left(k_t\right) = f'\left(k_t\right). \tag{1.2}$$

Notice that $w'\left(k_t\right) > 0$, and that $R'\left(k_t\right) < 0$.

Now, let's turn to the consumers. Individuals live for two periods and they are endowed with one unit of labor in each period. In their first period they are young and inelastically supply one unit of labor. In their second period they are old and care about leisure. Hence, old agents

[2] As one period represents about 35 years, it is reasonable to assume that the depreciation rate is 1.

make a trade-off between leisure and consumption. The portion of their old age devoted to leisure is called retirement. Let $c_{1,t}$ $\left(c_{2,t+1}\right)$ denote the consumption of the final good at date t (date $t+1$) by a representative young (old) agent born at t. Let $l_{t+1} \in [0,1]$ denote the labor supply by an old agent at time $t+1$. The length of time, $1-l_{t+1}$, is termed retirement and is endogenously determined. Preferences are given by a time-separable utility function:

$$U\left(c_{1,t}, c_{2,t+1}, l_{t+1}\right) \equiv u\left(c_{1,t}\right) + \beta\left[v\left(c_{2,t+1}\right) + \mu h\left(1-l_{t+1}\right)\right], \tag{1.3}$$

where β is a discount factor, and μ measures the weight of leisure in the second period of life. It is assumed that u, v and h are each twice continuously differentiable. Moreover, $u' > 0 > u''$, $v' > 0 > v''$ and $h' > 0 > h''$. For future reference, define the various Arrow-Pratt measures of relative risk aversion to be:

$$\Phi_u \equiv -\frac{c_1 u''(c_1)}{u'(c_1)};\ \Phi_v \equiv -\frac{c_2 v''(c_2)}{v'(c_2)};\ \Phi_l \equiv -\frac{(1-l)h''(1-l)}{h'(1-l)}. \tag{1.4}$$

1.2.2 Equilibrium conditions

There are three markets in this economy. Equilibrium in the labor market in period t, is given by $L_t = 1 + l_t$, as the young individuals supply one unit of labor, and the old individuals supply l_t. Since labor is homogeneous, the wage rate is the same for young and old at any point in time. A young agent born at date t inelastically supply one unit of labor in a competitive labor market and earns a wage w_t, and when old, the individual chooses how much labor to supply at the then current wage, w_{t+1}.

In the capital market, the aggregate saving of the young in any period, becomes the start-of-period capital for the next period. Hence, the equilibrium condition in the capital market is given by $K_{t+1} = s_t$, where s_t is aggregate savings. Using the definition of k we can rewrite this as $k_{t+1}L_{t+1} = s_t$. Moreover, by combining this with the equilibrium condition on the labor market, equilibrium in the capital market becomes:

$$(1 + l_{t+1}) k_{t+1} = s_t. \tag{1.5}$$

Equation (1.5) relates capital intensity, savings and optimal labor supply by the elderly. According to Walras law in period t, equilibrium in the labor market and in the capital market implies equilibrium in the output market, which is given by $Y_t = c_{1,t} + c_{2,t} + K_{t+1}$.

1.2.3 Pensions

The PAYG pension scheme is run by the government. At any date t, it levies a payroll tax τ_t on each young and old worker to finance a lump-sum transfer B_t to each of the currently retired. In the model economy, retirement is imposed as a pre-condition to receive pensions. This specification fits well with current practice in a number of countries where there are double taxes on continued work: the payroll tax $\tau_t w_t l_t$ and the forgone pension benefits $B_t l_t$. In every period, the government balances the tax income and the social security benefit in a PAYG program. The balanced budget scheme is thus given by:

$$(1 - l_t) B_t = \tau_t w_t + \tau_t w_t l_t. \tag{1.6}$$

The left hand side represents the pension benefits paid to current retirees and the right hand side captures the total social security tax revenue, $\tau_t w_t$ from young agents and $\tau_t w_t l_t$ from current old workers.

Here, the old agent contributes to his own pension. This makes the formulation a bit different from the standard one where only the current young contribute to the pension for the current, retired old. Also, as all this is happening within the same period, the agent does not earn any interest on his old-age contributions.

To understand how the PAYG social security system affects the capital-labor ratio and long-run welfare, it is crucial to understand how the consumers behave. In the next section we therefore solve the agent's problem and explore how the agent's optimal savings and labor supply respond to ceteris-paribus changes in the PAYG system.

1.3 Individual's Problem

Recall that an individual born in period t inelastically supply one unit of labor. The resulting labor income is allocated to first period consumption, savings and tax contributions. Thus:

$$c_{1,t} = (1-\tau_t) w_t - s_t. \tag{1.7}$$

In period $t+1$, the individual can consume out of accumulated savings, the income earned while working, and the pension benefit while retired. Accordingly:

$$c_{2,t+1} = R_{t+1} s_t + (1-\tau_{t+1}) w_{t+1} l_{t+1} + B_{t+1}(1-l_{t+1}). \tag{1.8}$$

The problem of an agent born at time t, taking $\tau_t, \tau_{t+1}, w_t, w_{t+1}, R_{t+1}$ and B_{t+1} as given, is written as:

$$\max_{c_{1,t}\geq 0, c_{2,t+1}\geq 0, l_{t+1}\in[0,1]} U(c_{1,t}, c_{2,t+1}, l_{t+1}) = u(c_{1,t}) + \beta\left[v(c_{2,t+1}) + \mu h(1-l_{t+1})\right],$$

subject to (1.7) and (1.8). It is assumed that the individuals do not perceive a link between their social security contributions and the benefits they will receive.

By inserting the budget constraints in (1.7) and (1.8) into the utility function, the problem becomes:

$$\max_{s_t, l_{t+1}} U(\cdot) = u((1-\tau_t)w_t - s_t)$$
$$+\beta\left[v\left(R_{t+1}s_t + (1-\tau_{t+1})w_{t+1}l_{t+1} + B_{t+1}(1-l_{t+1})\right) + \mu h(1-l_{t+1})\right].$$

Assuming interior solutions, the first order conditions with respect to savings and labor supply are given by:

$$s_t : -u'(c_{1,t}) + \beta R_{t+1} v'(c_{2,t+1}) = 0, \tag{1.9}$$

and

$$l_{t+1} : \left[w_{t+1}(1-\tau_{t+1}) - B_{t+1}\right] v'(c_{2,t+1}) - \mu h'(1-l_{t+1}) = 0. \tag{1.10}$$

The second order conditions are satisfied by the quasi-concavity assumption of $U(\cdot)$.

Equation (1.9) implies that the marginal rate of substitution of young-age consumption for old-age consumption equals the discount factor, $\frac{1}{R_{t+1}}$. This is the same as in the standard Diamond model. Equation (1.10) implies that the marginal rate of substitution between old-age consumption and leisure equals the price of lengthening retirement, $w_{t+1}(1-\tau_{t+1}) - B_{t+1}$. Solving (1.9) and (1.10), we can write the optimal savings and labor supply functions as

$$s_t = s\left(w_t(1-\tau_t), w_{t+1}(1-\tau_{t+1}), R_{t+1}, B_{t+1}\right); l_{t+1} = l\left(w_t(1-\tau_t), w_{t+1}(1-\tau_{t+1}), R_{t+1}, B_{t+1}\right).$$

We derive several results regarding individual's behavior in the following lemma.

Lemma 1.1 *(a)* $0 < s_1 < 1, s_2 < 0$*, and the signs of* s_3 *and* s_4 *are ambiguous.*

(b) $l_1 < 0, l_3 < 0, l_4 < 0$*, and the sign of* l_2 *is ambiguous.*

The proof of Lemma 1.1 is in Appendix 1.A. Let us briefly look at the intuitions behind the results in Lemma 1.1.

(i) The effects of current net wage rate $w_t(1-\tau_t)$: the young-age labor supply is fixed at one, thus a change in the current net wage rate only causes an income effect. Accordingly, with a higher income in young age, agents save more when young, and work less when old.

(ii) The effects of future net wage rate $w_{t+1}(1-\tau_{t+1})$: a decrease in the old-age net wage reduce future income. Since the individual has a preference for consumption smoothing, this causes the agent to save more. However, when the second period labor supply is endogenous, the effect of a change in the old-age net wage is not clear-cut. A reduction in the net wage will according to the substitution effect imply less labor supply, but according to the income effect imply an increase in labor supply.

(iii) The effects of interest rate R_{t+1}: changing the interest rate causes both income and substitution effects on savings. An increase in the interest rate will according to the substitution effect increase savings, but according to the income effect reduce savings. Hence, the total effect is ambiguous. For the labor supply, an increase in the interest rate increases the lifetime income and the income effect will reduce the labor supply in the old age.

(iv) The effects of pension level B_{t+1}: because agents can get higher pensions after retirement, they have less incentive to work and thus will retire earlier. But even though they work less in the second period, the old age wealth will increase, so the agents save less.

Lemma 1.1 shows that for several cases, changes in the interest rate and the net wage, will cause substitution and income effects that may work in opposite directions. Due to these unclear effects, the following assumptions will be made in the remainder of the paper. Firstly, the substitution effect of an interest rate increase dominates the income effect. As seen later, this implies that optimal saving is increasing in its return. Secondly, the substitution effect of an increase in the net wage dominates the income effect. It implies that optimal old-age labor supply is rising in the second period wage and falling in the second period tax rate.[3]

These assumptions are ensured by the following restrictions for any $\tau \in [0,1]$ at any period:

Assumption 1.1 $v'(\cdot) + Rsv''(\cdot) \geq 0$.

Assumption 1.2 $v'(\cdot) + v''(\cdot)(l+\tau)w \geq 0$.

Thus far we have studied the first order responses of savings and labor supply. It is clear that a PAYG pension program can affect the capital-labor ratio through several channels, and we can distinguish between direct and indirect effects. In each period, the changes in tax rates and pension levels will change young agents' savings and old agents' labor supply -- these are direct effects. There are also indirect effects through factor prices. As the capital-labor ratio changes, the wage rates and the interest rate will change, which in turn affect the agents' behavior. Moreover, there is always a link between social security taxes and the benefits if the government runs a balanced budget constraint. With this in mind, we proceed to study the general equilibrium responses in the next section.

1.4 General Equilibrium

In this section, we study the effects of increasing the size of a PAYG pension system in a general equilibrium. The government's budget constraint is given by equation (1.6). Solving for B, the one-period forward formulation becomes:

$$B_{t+1} = \frac{\tau_{t+1} w_{t+1} (1 + l_{t+1})}{1 - l_{t+1}}. \tag{1.11}$$

[3] These assumptions are commonly made and supported by several empirical studies (see Boskin 1978, Ashenfelter and Heckman, 1974).

If the government runs a balanced budget, a change in old-age wage rate w_{t+1} or old-age tax rate τ_{t+1} will affect savings and labor supply through the pension channel as well. Notice that although τ_t, w_t and R_{t+1} are not shown in (1.11), they will also affect the benefits through old-age labor supply. For example, as young-age wage rate increases, savings are raised and old-age labor supply is reduced (Lemma 1.1). The latter causes the benefits to decrease because the government collects less tax from the old workers.

Henceforward, we assume a constant tax policy, $\tau_t = \tau$ for all t. The capital accumulation equation (1.5), together with the factor market equilibrium conditions (1.1) and (1.2), implies the equilibrium law of motion for the capital-labor ratio, given $k_0 > 0$:

$$k_{t+1}\left[1+\hat{l}\left(\tau, w(k_t), w(k_{t+1}), R(k_{t+1})\right)\right] = \hat{s}\left(\tau, w(k_t), w(k_{t+1}), R(k_{t+1})\right). \tag{1.12}$$

where $\hat{s}(\tau, w_t, w_{t+1}, R_{t+1})$ and $\hat{l}(\tau, w_t, w_{t+1}, R_{t+1})$, denotes the general equilibrium savings and labor supply respectively. By using (1.9), (1.10) and (1.11), optimal savings and labor supply in general equilibrium, can be solved from the following two equations by using:

$$-u'\left((1-\tau)w_t - \hat{s}_t\right) + \beta R_{t+1} v'\left(R_{t+1}\hat{s}_t + w_{t+1}\hat{l}_{t+1} + \tau w_{t+1}\right) = 0, \tag{1.13}$$

$$w_{t+1}\left(1 - \frac{2\tau}{1-\hat{l}_{t+1}}\right) v'\left(R_{t+1}\hat{s}_t + w_{t+1}\hat{l}_{t+1} + \tau w_{t+1}\right) - \mu h'\left(1-\hat{l}_{t+1}\right) = 0. \tag{1.14}$$

For future use, we collect some results in Lemma 1.2:

Lemma 1.2 *Given assumptions 1.1 and 1.2,*

(a) $\hat{s}_{w_t} > 0, \hat{s}_{w_{t+1}} < 0, \hat{s}_{R_{t+1}} > 0$, *and the sign of* $\hat{s}_\tau$ *is ambiguous.*

(b) $\hat{l}_{w_t} < 0, \hat{l}_{w_{t+1}} > 0, \hat{l}_{R_{t+1}} < 0$, *and the sign of* $\hat{l}_\tau$ *is ambiguous.*

The proof of Lemma 1.2 is in Appendix 1.B.

In the remainder of the paper, we are more interested in a special case -- the steady-state equilibrium. Specifically, for a given τ, the capital-labor ratio k is stationary and satisfies:

$$k\left[1+\hat{l}\left(\tau,w(k),R(k)\right)\right]=\hat{s}\left(\tau,w(k),R(k)\right). \tag{1.15}$$

It is assumed that a steady-state k exists. The conditions needed to ensure existence are straightforward and discussed in Nourry (2001).

1.4.1 Neutrality of the PAYG system

Recall that the purpose of this paper is to study the effect of a PAYG system on capital intensity and welfare. With respect to capital intensity, we are particularly interested in the possible neutrality of a PAYG system. To proceed, apply the law of motion for the capital-labor ratio, and examine the overall effects of τ on the steady-state k. Equation (1.15) gives us

$$\frac{dk}{d\tau}=\frac{\hat{s}_\tau-k\hat{l}_\tau}{1+\hat{l}+k\left(\hat{l}_w w_k+\hat{l}_R R_k\right)-\left(\hat{s}_w w_k+\hat{s}_R R_k\right)}, \tag{1.16}$$

where $\hat{s}_w\equiv\hat{s}_{w_t}+\hat{s}_{w_{t+1}}\Big|_{\text{steady-state}}$, and $\hat{l}_w\equiv\hat{l}_{w_t}+\hat{l}_{w_{t+1}}\Big|_{\text{steady-state}}$.

According to Lemma 1.2, the sign of (1.16) is indeterminate in this general set up. However, we can determine the sign of the denominator, by applying Samuelson's correspondence principle. This principle suggests a link between the stability properties of a steady-state and its comparative statics properties.

The stability properties of a steady-state equilibrium are studied in the following manner. Taking the derivative with respect to k_t on both sides of (1.12) and reorganizing yields:

$$\frac{dk_{t+1}}{dk_t}=\frac{\left(\hat{s}_{w_t}-k_{t+1}\hat{l}_{w_t}\right)w_k\left(k_t\right)}{1+\hat{l}_{t+1}+k_{t+1}\left(\hat{l}_{w_{t+1}}w_k\left(k_{t+1}\right)+\hat{l}_{R_{t+1}}R_k\left(k_{t+1}\right)\right)-\left(\hat{s}_{w_{t+1}}w_k\left(k_{t+1}\right)+\hat{s}_{R_{t+1}}R_k\left(k_{t+1}\right)\right)}. \tag{1.17}$$

According to Lemma 1.2, both the numerator and denominator of (1.17) are positive. Around a locally stable steady-state, we have $\left.\frac{dk_{t+1}}{dk_t}\right|_k < 1$, which gives us

$$1+\widehat{l}+k\left(\left(\widehat{l}_{w_t}+\widehat{l}_{w_{t+1}}\right)w_k+\widehat{l}_{R_{t+1}}R_k\right)-\left.\left(\left(\widehat{s}_{w_t}+\widehat{s}_{w_{t+1}}\right)w_k+\widehat{s}_{R_{t+1}}R_k\right)\right|_k>0,$$

and further as

$$1+\widehat{l}+k\left(\widehat{l}_w w_k+\widehat{l}_R R_k\right)-\left.\left(\widehat{s}_w w_k+\widehat{s}_R R_k\right)\right|_k>0.$$

We can then establish the following lemmas.

Lemma 1.3 *At a (locally) stable steady-state, and given assumptions 1.1 and 1.2, the following holds:*

$$1+\widehat{l}+k\left(\widehat{l}_w w_k+\widehat{l}_R R_k\right)-\left(\widehat{s}_w w_k+\widehat{s}_R R_k\right)>0.$$

Hence, the denominator of (1.16) is positive.

Lemma 1.4 $\frac{dk}{d\tau}$ *has the same sign as* $\widehat{s}_\tau - k\widehat{l}_\tau$.

So to evaluate the effects of a PAYG pension system and the possibility of neutrality, it is necessary to determine the sign of ($\widehat{s}_\tau - k\widehat{l}_\tau$). As stated earlier, the signs of $\widehat{s}_\tau$ and $\widehat{l}_\tau$ are not determinate, which implies that even at a stable steady-state, we cannot sign $dk/d\tau$.

At this stage, it is instructive to pause briefly and consider what is known from a standard Diamond model. In such a model, where young agents work one unit of time and old agents are retired the entire second period, a well-known result is that a PAYG system will always reduce the incentive to save and thereby reduce capital accumulation at a stable steady-state (see Blanchard and Fischer, 1989). If we retain the same notation but set $l \equiv 0$, the felicities will only depend on consumptions when young and old. We can then show that (1.16) reduces to:

$$\frac{dk}{d\tau} = \frac{\widehat{s}_\tau}{1-\left(\widehat{s}_w w_k + \widehat{s}_R R_k\right)}.$$

Using the first order condition

$$-u'\left(\left(1-\tau\right)w-\widehat{s}\right)+\beta R v'\left(R\widehat{s}+\tau w\right)=0,$$

we can derive $\widehat{s}_\tau < 0$ and $\widehat{s}_w < 0$. Besides, under Assumption 1.1, $\widehat{s}_R > 0$ and since $w_k > 0$ and $R_k < 0$, we get

$$sign\left(\frac{dk}{d\tau}\right) = sign(\widehat{s}_\tau) < 0.$$

The implication is clear, a PAYG system is never neutral in the standard Diamond model.

But in the presence of endogenous retirement, a PAYG system will not only affect saving but also the old-age labor supply. And since there are countervailing effects in this set-up, it is by no means clear-cut how the capital-labor ratio will be affected. All sorts of possibilities, including full neutrality, may emerge, as will be evident from the specific example presented in the next section.

1.4.2 CES utility

In this section we look at the special case with a CES utility function. With this specification, we get analytical results for all feasible τ. The utility functions are:

$$u\left(x\right)=v\left(x\right)=h\left(x\right)=\frac{1}{1-\gamma}x^{1-\gamma}.$$

Then $\Phi_u = \Phi_v = \Phi_l = \gamma > 0$ where γ is the elasticity of marginal utility. Our purpose is to evaluate $\widehat{s}_\tau - k\widehat{l}_\tau$. Using the expressions for $\widehat{s}_\tau$ and $\widehat{l}_\tau$ evaluated at the steady-state, and combined with the CES utility function, we obtain (see Appendix 1.C):

$$\left.\widehat{s}_{\tau}\right|_{\text{steady-state}}=\frac{\gamma\beta R\left(wv'\right)^{2}}{c_{2}}\frac{\frac{-2\tau}{\left(1-\widehat{l}\right)^{2}}\left(\beta R\right)^{\frac{1}{\gamma}}+\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}-\left[\left(\beta R\right)^{\frac{1}{\gamma}}\left(\frac{w}{c_{2}}+\frac{1}{1-\widehat{l}}\right)+\frac{1}{1-\widehat{l}}\right]\left(1-\frac{2\tau}{1-\widehat{l}}\right)\gamma}{D_{2}} \tag{1.18}$$

and

$$\left.\widehat{l}_{\tau}\right|_{\text{steady-state}}=\frac{\gamma\beta R\left(wv'\right)^{2}}{c_{2}}\frac{\frac{2}{1-\widehat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}+\left(R-1\right)\left(1-\frac{2\tau}{1-\widehat{l}}\right)\frac{\left(\beta R\right)^{\frac{1}{\gamma}}}{c_{2}}\gamma}{D_{2}} \tag{1.19}$$

where D_2 is defined in Appendix 1.B. With CES utility functions, one of the first order conditions in steady-state is:

$$\left(1-\tau\right)w-\widehat{s}=\left(\beta R\right)^{-\frac{1}{\gamma}}\left(R\widehat{s}+w\widehat{l}+\tau w\right),$$

from which we can solve

$$\widehat{s}=\frac{\left(1-\tau\right)w-\left(\beta R\right)^{-\frac{1}{\gamma}}\left(w\widehat{l}+\tau w\right)}{\left(\beta R\right)^{-\frac{1}{\gamma}}R+1}.$$

Thus the capital-labor ratio is:

$$k=\frac{\widehat{s}}{1+\widehat{l}}=\frac{\left(1-\tau\right)w-\left(\beta R\right)^{-\frac{1}{\gamma}}\left(w\widehat{l}+\tau w\right)}{\left(\left(\beta R\right)^{-\frac{1}{\gamma}}R+1\right)\left(1+\widehat{l}\right)} \tag{1.20}$$

By combining this with the resource constraint of the economy, we derive the following proposition.

Proposition 1.1 *Under CES utility with* $\Phi_u=\Phi_v=\Phi_l=\gamma>0$,

$$\operatorname{sign}\frac{dk}{d\tau}=\operatorname{sign}\left(1-\gamma\right)\begin{cases}>0 & \text{if} \quad \gamma<1\\ =0 & \text{if} \quad \gamma=1\\ <0 & \text{if} \quad \gamma>1\end{cases}.$$

The proof of Proposition 1.1 is in Appendix 1.D. Thus, the effect of the PAYG contribution rate only depends on the elasticity of marginal utility. Notice that this proposition applies for the whole range of τ .

As shown above, in the standard Diamond model, a PAYG system is never neutral. An increase in the size of the system necessarily reduces capital formation. This is the original concern with PAYG systems. Here, if endogenous retirement is taken into account, we find introducing a PAYG pension can reduce the capital stock, increase the capital stock, or even be neutral. With CES utility, the effect of the PAYG contribution rate would rely totally on the size of γ relative to 1. In the often studied case with log utility, $\gamma = 1$, and it follows that $dk / d\tau = 0$. The implication is startling: if preferences conform to the commonly studied logarithmic utility, introducing a PAYG system is exactly neutral.

Another unconventional result can also be derived from Proposition 1.1. Notice that a sufficient condition for Assumption 1 and Assumption 2 to hold is $\Phi_v \leq 1$. Hence, since $\Phi_v = \gamma \leq 1$, one can derive $dk / d\tau \geq 0$ with no additional constraints on the preferences than just CES utility. Rather than being a source of concern, in this case, a PAYG system can even help to expand capital formation.

In Diamond (1965), a PAYG system reduces the capital stock and via that channel, reduces steady-state welfare if the economy is not overaccumulating capital to begin with. And since most countries do not overaccumulate capital, the PAYG system weakens the welfare. The potential to overturn this line of logic is now possible. In the model with endogenous retirement, under some circumstances, a PAYG system can help to expand capital formation. If the economy is initially dynamically efficient, this helps to cause the rate of return on capital to converge to the Golden Rule level, which increases the long-run welfare of the individuals. However, from equation

(1.14), it is apparent that an increase in the size of the pension program will distort, causing the marginal rate of substitution between old-age consumption and leisure (retirement) to diverge from the wage rate, and reducing welfare. Whether the total effect on welfare is positive depends on whether the increased dynamic efficiency exceeds the static inefficiency it brings about, and that is the subject of the next section.

1.5 Welfare

Here we study the effect of implementing a pension system on the long-run (steady-state) equilibrium level of utility enjoyed by a representative agent. In the standard Diamond model, a PAYG system can never be Pareto-improving in a dynamically efficient economy. The initial generation gains, but future generations lose. This is because for future generations the rate of return on the taxes they pay when young is 1 (assuming no population growth), whereas their own saving via capital would have generated a return of $R>1$. Thus each future generation incurs a loss because it receives a return 1 on the social security taxes which is less than the return R it would earn by investing those funds in capital. In the standard Diamond model, therefore, whether the PAYG system is desirable depends only on whether the economy is dynamically efficient or not. Once endogenous retirement is introduced, matters are far more complicated because of the presence of another effect: agents modify their old-age labor supply in response to the pension program. In this section, we study this combined effect on welfare.

The government is assumed to be benevolent. It chooses τ to maximize the utility of an agent in a steady-state equilibrium. The government's problem is written as:

$$\max_{\tau} U(\tau) = u\left[c_1(\tau)\right] + \beta v\left[c_2(\tau)\right] + \beta\mu h\left[1 - l(\tau)\right] \tag{1.21}$$

with

$$c_1(\tau)=(1-\tau)w[k(\tau)]-\widehat{s}\{\tau,w[k(\tau)],R[k(\tau)]\},$$

$$c_2(\tau)=R[k(\tau)]\widehat{s}\{\tau,w[k(\tau)],R[k(\tau)]\}+w[k(\tau)]\widehat{l}\{\tau,w[k(\tau)],R[k(\tau)]\}+\tau w[k(\tau)],$$

$$l(\tau)=\widehat{l}\{\tau,w[k(\tau)],R[k(\tau)]\}.$$

Differentiating (1.21) with respect to τ and reorganizing, we get:

$$\frac{dU}{d\tau}=-wu'(c_1)+\beta v'(c_2)w+\left[(1-\tau)w_k u'(c_1)+\beta v'(c_2)\left(R_k\widehat{s}+w_k\widehat{l}+\tau w_k\right)\right]\frac{dk}{d\tau}$$
$$+\left[\beta v'(c_2)R-u'(c_1)\right]\frac{d\widehat{s}}{d\tau}+\beta\left[v'(c_2)w-\mu h'\left(1-\widehat{l}\right)\right]\frac{d\widehat{l}}{d\tau},$$

where

$$\frac{d\widehat{s}}{d\tau}=\widehat{s}_\tau+\left(\widehat{s}_w w_k+\widehat{s}_R R_k\right)\frac{dk}{d\tau},$$

$$\frac{d\widehat{l}}{d\tau}=\widehat{l}_\tau+\left(\widehat{l}_w w_k+\widehat{l}_R R_k\right)\frac{dk}{d\tau}.$$

From the first order conditions (1.13) and (1.14), we know that $u'(c_1)=\beta Rv'(c_2)$ and $w\left(1-\frac{2\tau}{1-\widehat{l}}\right)v'(c_2)=\mu h'\left(1-\widehat{l}\right)$. Also, $w_k=-kR_k$ and $\widehat{s}=(1+\widehat{l})k$. By exploiting these relations, the effect of τ can be written as:

$$\frac{dU}{d\tau}=\beta w(1-R)v'(c_2)+(1-\tau)k(1-R)\beta R_k v'(c_2)\frac{dk}{d\tau}+\beta\frac{2\tau}{1-\widehat{l}}wv'(c_2)\frac{d\widehat{l}}{d\tau}.$$

Evaluated near $\tau=0$, it all boils down to a simple-looking condition:

$$\left.\frac{dU}{d\tau}\right|_{\tau=0}=\underbrace{\beta(1-R)v'(c_2)}_{(1)}\left.\underbrace{\left(w-w_k\frac{dk}{d\tau}\right)}_{(2)}\right|_{\tau=0} \tag{1.22}$$

In a dynamically efficient economy, factor (1) is negative, while factor (2) is indeterminate. The sign of factor (2) is partly determined by Proposition 1.1. Accordingly, whether a PAYG system is desirable or not, does not solely depend on dynamic efficiency or inefficiency, it also depends on how k responds, i.e., on $dk/d\tau$. Indeed, the possibility arises that there may be long-run welfare gain from introducing a PAYG pension. After all, in a dynamically efficient economy, the capital-labor ratio level is lower compared to its Golden Rule level. As stated earlier, if the pension system can increase capital-labor ratio, and the increased dynamic efficiency exceeds the static inefficiency it brings about, the total effect on long-run welfare will be positive.

Based on Proposition 1.1, and equation (1.22), we deduce the following proposition.

Proposition 1.2 *For a dynamically efficient* $(R>1)$ *economy with endogenous retirement and CES utility,*

$$\left.\frac{dU}{d\tau}\right|_{\tau=0} = \underbrace{\beta(1-R)v'(c_2)}_{(1)} \left.\underbrace{\left(w - w_k \frac{dk}{d\tau}\right)}_{(2)}\right|_{\tau=0} \begin{cases} >0 & \text{if } \gamma<1 \text{ and } (2)>0 \\ \leq 0 & \text{otherwise} \end{cases} \qquad (1.23)$$

That is:

If $\gamma \geq 1$, *there is no long-run welfare case for introducing a PAYG pension;*

If $\gamma < 1$, *introducing a PAYG pension may be beneficial in terms of long-run welfare. It depends on the sign of factor (2).*

This proposition contains several results. Firstly, there is no long-run welfare gain from introducing a PAYG pension if $\gamma \geq 1$. This is in contrast with the standard model without endogenous retirement where this would hold for any γ. Secondly, in a standard set up, if the pension is neutral in k, it is neutral in welfare; that is not the case here. Thirdly, it seems to be possible that welfare could even go up upon introduction of a PAYG system for $\gamma < 1$. In this case, the capital-labor ratio has to have increased. Andersen and Bhattacharya (2013) show in a model with endogenous young-age labor supply that there is no case for introducing a PAYG pension if

$\gamma \leq 1$ in a dynamically efficient economy (this is a special case of their lemma 8). However, if we allow endogenous labor supply in the second period instead of in the first period, the conclusion changes. The reason is that in their setup, if $\gamma \leq 1$, introducing a social security system will cause the capital-labor ratio to decrease. Thus, the role of the capital-labor ratio is very essential when measuring the welfare.

We close this section with a numerical example. Suppose

$$U = \tfrac{1}{1-\gamma} c_1^{1-\gamma} + \beta \left[\tfrac{1}{1-\gamma} c_2^{1-\gamma} + \mu \tfrac{1}{1-\gamma} (1-l)^{1-\gamma} \right],$$

$$F = A\left(\alpha K^{-\rho} + (1-\alpha) L^{-\rho}\right)^{-\frac{1}{\rho}}, \rho \geq -1.$$

With $\gamma = 0.5, \beta = 0.5, \mu = 1, \alpha = 0.5, \rho = 3$ and $A = 5.6$, Figure 1.1 plots the effects of introducing a PAYG pension on the economy. As Figure 1.1 illustrates, in an initially dynamically efficient economy, introducing a PAYG pension will increase the capital-labor ratio and increase the long-run welfare when τ is in the range of $(0, 8\%)$.[4]

1.6 Conclusions

A classic result in dynamic public economics states that a PAYG system will reduce capital intensity, and if the economy is dynamically efficient, the system will reduce welfare as well. This result can be shown in a standard Diamond OG model with exogenous labor supply. In this paper, we study these established results in a setting where the endogenous retirement is included. The relation between the pension system, and capital intensity and welfare, becomes significantly more complex. The reason is that a second channel emerges. It relates the pension system and labor

[4] When $\tau = 8\%$, we have $R = 3.4$. By considering the length of each period to be 35 years, it implies an annual interest rate of 3.6%.

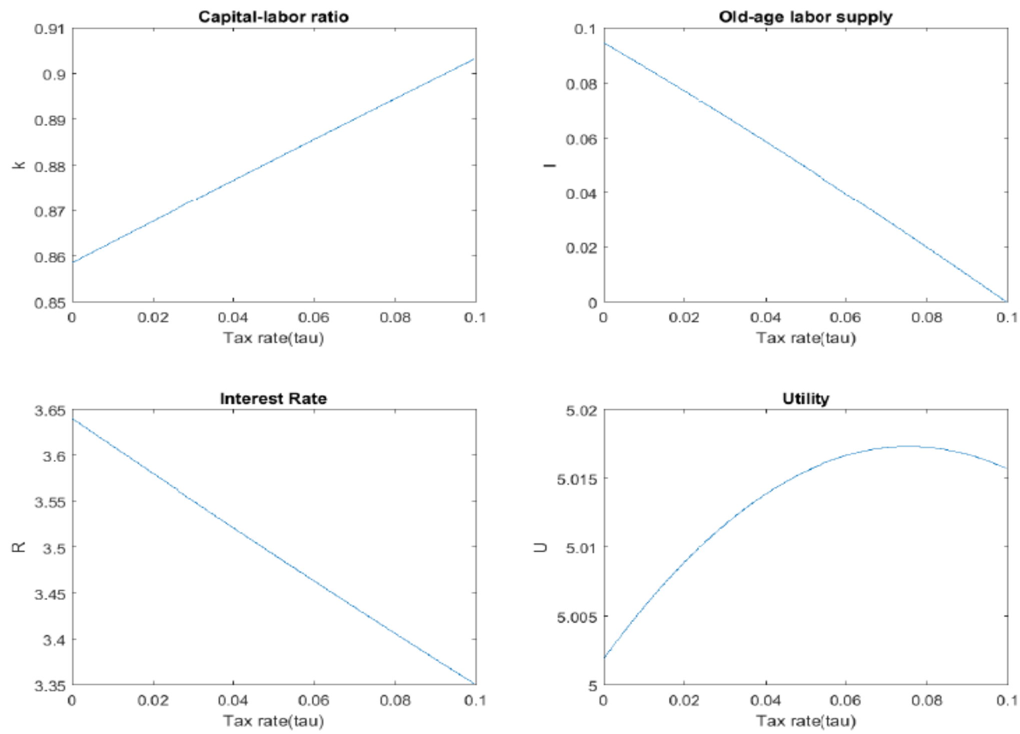

Figure 1.1. *The effects of introducing a PAYG pension*

supply, and thus the capital-labor ratio, along with the relation between savings, factor prices and capital accumulation. Within a general environment, we find that the effect of a PAYG pension system on the steady-state capital-labor ratio is not clear-cut. It depends on the magnitude and directions of the saving and labor supply responses to a change in the pension system.

To make the analysis and the results more tractable (and easier to follow), the model is specified with a CES utility function. In this specified environment, it is found that the effect of the PAYG system on the capital-labor ratio will only depend on the elasticity of substitution between consumptions and leisure relative to one. Specifically, in the logarithmic utility case, introducing a PAYG system is neutral in terms of capital-labor ratio.

We also look at the welfare effects of introducing a PAYG pension system. Unlike in the standard Diamond model where it only depends on whether the initial steady-state is dynamically efficient or inefficient, here it also depends on how the capital-labor ratio reacts. For an initially dynamically efficient economy, if the capital-labor ratio increases with the introduction of PAYG pensions, it may be welfare-enhancing, depending on whether the increased dynamic efficiency exceeds the static inefficiency. In short, in a dynamically efficient economy with endogenous retirement, a PAYG system may be desirable.

In conclusion, it is important that a caveat be recorded. The analysis in this paper has been entirely restricted to steady states. It is shown that the introduction of a PAYG system may be desirable from the perspective of the long run. Our justification for using steady-state welfare as the yardstick for judging desirability is simple: if a policy fails to generate long-run welfare gains, it is unlikely to be ever adopted. However, despite our demonstration of the steady-state welfare gains, it may be questioned whether such an arrangement can ever be implemented in the short run. After all, it is quite possible that some initial generations are hurt in the process of getting to that long-run outcome. There may be ways to leave the welfare of these generations unchanged even after the policy is introduced. Presumably there are mechanisms -- similar in spirit to ones studied in Rangel (2003) and Boldrin and Montes (2005) -- which allow some of the future welfare gains to be brought forward to compensate the initial generations. Such matters are worthy of future study.

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APPENDIX 1.A

PROOF OF LEMMA 1.1

Using the first order conditions (1.9) and (1.10), we get

$$\begin{pmatrix} u''(\cdot)+\beta R_{t+1}^2 v''(\cdot) & \beta R_{t+1}\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]v''(\cdot) \\ R_{t+1}\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]v''(\cdot) & \left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]^2 v''(\cdot)+\mu h''(\cdot) \end{pmatrix}\begin{pmatrix} \partial s_t \\ \partial l_{t+1} \end{pmatrix}$$

$$=-\begin{pmatrix} -u''(\cdot) \\ 0 \end{pmatrix}\partial\left[w_t(1-\tau_t)\right]-\begin{pmatrix} \beta R_{t+1} l_{t+1} v''(\cdot) \\ v'(\cdot)+\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]l_{t+1}v''(\cdot) \end{pmatrix}\partial\left[w_{t+1}(1-\tau_{t+1})\right]$$

$$-\begin{pmatrix} \beta v'(\cdot)+\beta R_{t+1} s_t v''(\cdot) \\ \left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]s_t v''(\cdot) \end{pmatrix}\partial R_{t+1}-\begin{pmatrix} \beta R_{t+1}(1-l_{t+1})v''(\cdot) \\ -v'(\cdot)+\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right](1-l_{t+1})v''(\cdot) \end{pmatrix}\partial B_{t+1}$$

Define the determinant of the coefficient matrix to be D. It is easy to show that

$$D=u''(\cdot)\left\{\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]^2 v''(\cdot)+\mu h''(\cdot)\right\}+\beta\mu R_{t+1}^2 v''(\cdot)h''(\cdot)>0$$

Using Cramer's rule and (9) (10), we get

$$\frac{\partial s_t}{\partial\left[w_t(1-\tau_t)\right]}=u''(\cdot)\frac{\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]^2 v''(\cdot)+\mu h''(\cdot)}{D}\in(0,1)$$

$$\frac{\partial s_t}{\partial\left[w_{t+1}(1-\tau_{t+1})\right]}=\mu\beta R_{t+1}v''(\cdot)\frac{h'(\cdot)-l_{t+1}h''(\cdot)}{D}<0$$

$$\frac{\partial s_t}{\partial R_{t+1}}=-\beta\mu\frac{\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]h'(\cdot)v''(\cdot)-\left[v'(\cdot)+R_{t+1}s_t v''(\cdot)\right]h''(\cdot)}{D}$$

$$\frac{\partial s_t}{\partial B_{t+1}}=-\mu\beta R_{t+1}h'(\cdot)v''(\cdot)\frac{\left[1+\frac{(1-l_{t+1})h''(\cdot)}{h'(\cdot)}\right]}{D}$$

$$\frac{\partial l_{t+1}}{\partial\left[w_t(1-\tau_t)\right]}=\frac{-R_{t+1}\left[w_{t+1}(1-\tau_{t+1})-B_{t+1}\right]u''(\cdot)v''(\cdot)}{D}<0$$

$$\frac{\partial l_{t+1}}{\partial\left[w_{t+1}\left(1-\tau_{t+1}\right)\right]} = -\frac{v'(\cdot)u''(\cdot)+R_{t+1}u'(\cdot)v''(\cdot)+\left[w_{t+1}\left(1-\tau_{t+1}\right)-B_{t+1}\right]l_{t+1}u''(\cdot)v''(\cdot)}{D}$$

$$\frac{\partial l_{t+1}}{\partial R_{t+1}} = \frac{\left[w_{t+1}\left(1-\tau_{t+1}\right)-B_{t+1}\right]\left[u'(\cdot)-s_t u''(\cdot)\right]v''(\cdot)}{D} < 0$$

$$\frac{\partial l_{t+1}}{\partial B_{t+1}} = \frac{v'(\cdot)u''(\cdot)+R_{t+1}u'(\cdot)v''(\cdot)-\left[w_{t+1}\left(1-\tau_{t+1}\right)-B_{t+1}\right]\left(1-l_{t+1}\right)u''(\cdot)v''(\cdot)}{D} < 0$$

APPENDIX 1.B

PROOF OF LEMMA 1.2

The proof of Lemma 1.2 is very similar to that of Lemma l.1. Using the first order conditions (1.13) and (1.14), we get

$$\begin{pmatrix} \left[u''(\cdot)+\beta R_{t+1}^2 v''(\cdot)\right] & \beta R_{t+1} w_{t+1} v''(\cdot) \\ R_{t+1} w_{t+1}\left(1-\frac{2\tau}{1-\hat{l}_{t+1}}\right)v''(\cdot) & \begin{bmatrix} \frac{-2\tau}{\left(1-\hat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \\ \left(1-\frac{2\tau}{1-\hat{l}_{t+1}}\right) w_{t+1}^2 v''(\cdot)+\mu h''(\cdot) \end{bmatrix} \end{pmatrix} \begin{pmatrix} \partial \hat{s}_t \\ \partial \hat{l}_{t+1} \end{pmatrix}$$

$$= -\begin{pmatrix} -\left(1-\tau\right)u''(\cdot) \\ 0 \end{pmatrix}\partial w_t - \begin{pmatrix} \beta R_{t+1}\left(\hat{l}_{t+1}+\tau\right)v''(\cdot) \\ \left(1-\frac{2\tau}{1-\hat{l}_{t+1}}\right)\left[v'(\cdot)+w_{t+1}\left(\hat{l}_{t+1}+\tau\right)v''(\cdot)\right] \end{pmatrix}\partial w_{t+1}$$

$$-\begin{pmatrix} \beta\left[v'(\cdot)+R_{t+1}\hat{s}_t v''(\cdot)\right] \\ \left(1-\frac{2\tau}{1-\hat{l}_{t+1}}\right) w_{t+1}\hat{s}_t v''(\cdot) \end{pmatrix}\partial R_{t+1} - \begin{pmatrix} w_t u''(\cdot)+\beta R_{t+1} w_{t+1} v''(\cdot) \\ w_{t+1}\left[\frac{-2}{1-\hat{l}_{t+1}} v'(\cdot)+\left(1-\frac{2\tau}{1-\hat{l}_{t+1}}\right) w_{t+1} v''(\cdot)\right] \end{pmatrix}\partial \tau$$

Define the determinant of the coefficient matrix to be D_2. It is easy to show that

$$D_2 = u''(\cdot)\left[\frac{-2\tau}{\left(1-\widehat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) w_{t+1}^2 v''(\cdot) + \mu h''(\cdot)\right]$$

$$+\beta R_{t+1}^2 v''(\cdot)\left[\frac{-2\tau}{\left(1-\widehat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \mu h''(\cdot)\right] > 0$$

Using Cramer's rule, we get

$$\widehat{s}_{w_t} = \frac{(1-\tau)u''(\cdot)\left[\frac{-2\tau}{\left(1-\widehat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) w_{t+1}^2 v''(\cdot) + \mu h''(\cdot)\right]}{D_2} > 0$$

$$\widehat{s}_{w_{t+1}} = \beta R_{t+1} v''(\cdot) \frac{-\left(\widehat{l}_{t+1}+\tau\right)\left[\frac{-2\tau}{\left(1-\widehat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \mu h''(\cdot)\right] + \left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) w_{t+1} v'(\cdot)}{D_2} < 0$$

$$\widehat{s}_{R_{t+1}} = \beta \frac{-\left[v'(\cdot) + R_{t+1}\widehat{s}_t v''(\cdot)\right]\left[\frac{-2\tau}{\left(1-\widehat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \mu h''(\cdot)\right] - \left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) w_{t+1}^2 v''(\cdot) v'(\cdot)}{D_2} > 0$$ [5]

$$\widehat{s}_{\tau} = \frac{-w_t u''(\cdot)\left[\frac{-2\tau}{\left(1-\widehat{l}_{t+1}\right)^2} w_{t+1} v'(\cdot) + \left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) w_{t+1}^2 v''(\cdot) + \mu h''(\cdot)\right] - \beta R_{t+1} w_{t+1} v''(\cdot)\left[\left(1-\frac{\tau}{1-\widehat{l}_{t+1}}\right)\frac{2}{1-\widehat{l}_{t+1}} w_{t+1} v'(\cdot) + \mu h''(\cdot)\right]}{D_2}$$

$$\widehat{l}_{w_t} = \frac{-(1-\tau)\left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) R_{t+1} w_{t+1} u''(\cdot) v''(\cdot)}{D_2} < 0$$

$$\widehat{l}_{w_{t+1}} = \left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right)\frac{-\left[v'(\cdot) + w_{t+1}\left(\widehat{l}_{t+1}+\tau\right) v''(\cdot)\right] u''(\cdot) - \beta R_{t+1}^2 v'(\cdot) v''(\cdot)}{D_2} > 0$$ [6]

$$\widehat{l}_{R_{t+1}} = w_{t+1}\left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right) v''(\cdot) \frac{\beta R_{t+1} v'(\cdot) - \widehat{s}_t u''(\cdot)}{D_2} < 0$$

---

[5] This holds under assumption 1.

[6] This holds under assumption 2.

$$\widehat{l}_\tau = w_{t+1}\frac{\frac{2}{1-\widehat{l}_{t+1}}v'(\cdot)\left[u''(\cdot)+\beta R_{t+1}^2 v''(\cdot)\right]+\left(w_t R_{t+1}-w_{t+1}\right)\left(1-\frac{2\tau}{1-\widehat{l}_{t+1}}\right)u''(\cdot)v''(\cdot)}{D_2}$$

APPENDIX 1.C

OPTIMAL SAVINGS AND LABOR SUPPLY

Evaluated at a steady-state,

$$\widehat{s}_\tau\Big|_{\text{steady-state}} = w\frac{-u''\left[\frac{-2\tau}{(1-\widehat{l})^2}wv'+\left(1-\frac{2\tau}{1-\widehat{l}}\right)w^2v''+\mu h''\right]-\beta Rv''\left[\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}wv'+\mu h''\right]}{D_2}$$

$$\widehat{l}_\tau\Big|_{\text{steady-state}} = w\frac{\frac{2}{1-\widehat{l}}v'\left(u''+\beta R^2v''\right)+w(R-1)\left(1-\frac{2\tau}{1-\widehat{l}}\right)u''v''}{D_2}$$

Using first order conditions $u'=\beta Rv'$ and $w\left(1-\frac{2\tau}{1-\widehat{l}}\right)v'=\mu h'$, and using $\Phi_u=\Phi_v=\Phi_l=\gamma$, we obtain

$$\widehat{s}_\tau\Big|_{\text{steady-state}}$$

$$= w\frac{-\frac{c_1u''}{u'}\frac{u'}{c_1}\left[\frac{-2\tau}{(1-\widehat{l})^2}wv'+\left(1-\frac{2\tau}{1-\widehat{l}}\right)w^2\frac{c_2v''}{v'}\frac{v'}{c_2}+\mu\frac{(1-\widehat{l})h''}{h'}\frac{h'}{1-\widehat{l}}\right]-\beta R\frac{c_2v''}{v'}\frac{v'}{c_2}\left[\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}wv'+\mu\frac{(1-\widehat{l})h''}{h'}\frac{h'}{1-\widehat{l}}\right]}{D_2}$$

$$= \gamma w\frac{\frac{u'}{c_1}\left[\frac{-2\tau}{(1-\widehat{l})^2}wv'-\left(1-\frac{2\tau}{1-\widehat{l}}\right)w^2\gamma\frac{v'}{c_2}-\mu\gamma\frac{h'}{1-\widehat{l}}\right]+\beta R\frac{v'}{c_2}\left[\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}wv'-\mu\gamma\frac{h'}{1-\widehat{l}}\right]}{D_2}$$

$$= \gamma\beta R(wv')^2\frac{\frac{1}{c_1}\left[\frac{-2\tau}{(1-\widehat{l})^2}-\left(1-\frac{2\tau}{1-\widehat{l}}\right)w\gamma\frac{1}{c_2}-\gamma\frac{\left(1-\frac{2\tau}{1-\widehat{l}}\right)}{1-\widehat{l}}\right]+\frac{1}{c_2}\left[\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}-\gamma\frac{\left(1-\frac{2\tau}{1-\widehat{l}}\right)}{1-\widehat{l}}\right]}{D_2}$$

Also, because $u'=\beta Rv'$ and $w\left(1-\frac{2\tau}{1-\widehat{l}}\right)v'=\mu h'$, we have $(c_1)^{-\gamma}=\beta R(c_2)^{-\gamma}$ and $w\left(1-\frac{2\tau}{1-\widehat{l}}\right)(c_2)^{-\gamma}=\mu\left(1-\widehat{l}\right)^{-\gamma}$, which imply $c_1=(\beta R)^{-\frac{1}{\gamma}}c_2$ and $1-\widehat{l}=\left(\frac{w}{\mu}\right)^{-\frac{1}{\gamma}}\left(1-\frac{2\tau}{1-\widehat{l}}\right)^{-\frac{1}{\gamma}}c_2$.

$$\widehat{s}_\tau\Big|_{\text{steady-state}} = \frac{\gamma\beta R\left(wv'\right)^2}{c_2}\frac{\left(\beta R\right)^{\frac{1}{\gamma}}\left[\frac{-2\tau}{\left(1-\widehat{l}\right)^2}-\left(\frac{w}{c_2}+\frac{1}{1-\widehat{l}}\right)\left(1-\frac{2\tau}{1-\widehat{l}}\right)\gamma\right]+\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}-\gamma\frac{\left(1-\frac{2\tau}{1-\widehat{l}}\right)}{1-\widehat{l}}}{D_2}$$

$$= \frac{\gamma\beta R\left(wv'\right)^2}{c_2}\frac{\frac{-2\tau}{\left(1-\widehat{l}\right)^2}\left(\beta R\right)^{\frac{1}{\gamma}}+\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}-\left[\left(\beta R\right)^{\frac{1}{\gamma}}\left(\frac{w}{c_2}+\frac{1}{1-\widehat{l}}\right)+\frac{1}{1-\widehat{l}}\right]\left(1-\frac{2\tau}{1-\widehat{l}}\right)\gamma}{D_2}$$

Similarly,

$$\widehat{l}_\tau\Big|_{\text{steady-state}} = w\frac{\frac{2}{1-\widehat{l}}v'\left(\frac{c_1u''}{u'}\frac{u'}{c_1}+\beta R^2\frac{c_2v''}{v'}\frac{v'}{c_2}\right)+w\left(R-1\right)\left(1-\frac{2\tau}{1-\widehat{l}}\right)\frac{c_1u''}{u'}\frac{u'}{c_1}\frac{c_2v''}{v'}\frac{v'}{c_2}}{D_2}$$

$$= \gamma w\frac{\frac{2}{1-\widehat{l}}v'\left(-\frac{u'}{c_1}-\beta R^2\frac{v'}{c_2}\right)+\gamma w\left(R-1\right)\left(1-\frac{2\tau}{1-\widehat{l}}\right)\frac{u'}{c_1}\frac{v'}{c_2}}{D_2}$$

$$= \gamma\beta R\left(v'\right)^2 w\frac{\frac{2}{1-\widehat{l}}\left(-\frac{1}{c_1}-R\frac{1}{c_2}\right)+\gamma w\left(R-1\right)\left(1-\frac{2\tau}{1-\widehat{l}}\right)\frac{1}{c_1}\frac{1}{c_2}}{D_2}$$

$$= \frac{\gamma\beta R\left(wv'\right)^2}{c_2}\frac{\frac{2}{1-\widehat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}+\left(R-1\right)\left(1-\frac{2\tau}{1-\widehat{l}}\right)\frac{\left(\beta R\right)^{\frac{1}{\gamma}}}{c_2}\gamma}{D_2}.$$

APPENDIX 1.D

PROOF OF PROPOSITION 1.1

Using the expressions for $\widehat{s}_\tau$ and $\widehat{l}_\tau$, from (1.18) and (1.19), we can derive the following:

$$\widehat{s}_\tau - k\widehat{l}_\tau\Big|_{\text{steady -state}} = \frac{\gamma\beta R\left(wv'\right)^2}{c_2}\times$$

$$\frac{\frac{-2\tau}{\left(1-\widehat{l}\right)^2}\left(\beta R\right)^{\frac{1}{\gamma}}+\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}-\frac{2}{1-\widehat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}k-\left[\left(\beta R\right)^{\frac{1}{\gamma}}\left(\frac{w}{c_2}+\frac{1}{1-\widehat{l}}\right)+\frac{1}{1-\widehat{l}}+\left(R-1\right)\frac{\left(\beta R\right)^{\frac{1}{\gamma}}}{c_2}k\right]\left(1-\frac{2\tau}{1-\widehat{l}}\right)\gamma}{D_2}$$

Define

$$A = \frac{-2\tau}{\left(1-\widehat{l}\right)^2}\left(\beta R\right)^{\frac{1}{\gamma}}+\left(1-\frac{\tau}{1-\widehat{l}}\right)\frac{2}{1-\widehat{l}}-\frac{2}{1-\widehat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}k$$

$$B=\left[\left(\beta R\right)^{\frac{1}{\gamma}}\left(\frac{w}{c_2}+\frac{1}{1-\hat{l}}\right)+\frac{1}{1-\hat{l}}+\left(R-1\right)\frac{\left(\beta R\right)^{\frac{1}{\gamma}}}{c_2}k\right]\left(1-\frac{2\tau}{1-\hat{l}}\right)>0 \text{ if } R>1$$

Then

$$
\begin{aligned}
&A-B\\
&=\frac{-2\tau}{\left(1-\hat{l}\right)^2}\left(\beta R\right)^{\frac{1}{\gamma}}+\left(1-\frac{\tau}{1-\hat{l}}\right)\frac{2}{1-\hat{l}}-\frac{2}{1-\hat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}k\\
&-\left[\left(\beta R\right)^{\frac{1}{\gamma}}\left(\frac{w}{c_2}+\frac{1}{1-\hat{l}}\right)+\frac{1}{1-\hat{l}}+\left(R-1\right)\frac{\left(\beta R\right)^{\frac{1}{\gamma}}}{c_2}k\right]\left(1-\frac{2\tau}{1-\hat{l}}\right)\\
&=\frac{-2\tau}{\left(1-\hat{l}\right)^2}\left(\beta R\right)^{\frac{1}{\gamma}}+\left(1-\frac{\tau}{1-\hat{l}}\right)\frac{2}{1-\hat{l}}-\frac{1}{1-\hat{l}}\left(1-\frac{2\tau}{1-\hat{l}}\right)-\frac{2}{1-\hat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}k\\
&-\left[\left(\beta R\right)^{\frac{1}{\gamma}}\left(\frac{w}{c_2}+\frac{1}{1-\hat{l}}\right)+\left(R-1\right)\frac{\left(\beta R\right)^{\frac{1}{\gamma}}}{c_2}k\right]\left(1-\frac{2\tau}{1-\hat{l}}\right)\\
&=\frac{-2\tau}{\left(1-\hat{l}\right)^2}\left(\beta R\right)^{\frac{1}{\gamma}}+\frac{1}{1-\hat{l}}-\frac{2}{1-\hat{l}}\frac{-\left(\beta R\right)^{\frac{1}{\gamma}}-R}{w}k-\left(\frac{w+\left(R-1\right)k}{c_2}+\frac{1}{1-\hat{l}}\right)\left(\beta R\right)^{\frac{1}{\gamma}}\left(1-\frac{2\tau}{1-\hat{l}}\right)\\
&=\left(1-\left(\beta R\right)^{\frac{1}{\gamma}}\right)\frac{1}{1-\hat{l}}+\frac{2}{1-\hat{l}}\frac{\left(\beta R\right)^{\frac{1}{\gamma}}+R}{w}k-\frac{w+\left(R-1\right)k}{c_2}\left(\beta R\right)^{\frac{1}{\gamma}}\left(1-\frac{2\tau}{1-\hat{l}}\right)\\
&=\frac{1}{1-\hat{l}}\left[1-\left(\beta R\right)^{\frac{1}{\gamma}}+2\frac{\left(\beta R\right)^{\frac{1}{\gamma}}+R}{w}k-\frac{1-\hat{l}}{c_2}\left[w+\left(R-1\right)k\right]\left(\beta R\right)^{\frac{1}{\gamma}}\left(1-\frac{2\tau}{1-\hat{l}}\right)\right]
\end{aligned}
$$

With CES Utility functions, one of the first order conditions in the steady state is

$$\left(1-\tau\right)w-\hat{s}=\left(\beta R\right)^{-\frac{1}{\gamma}}\left(R\hat{s}+w\hat{l}+\tau w\right)$$

from which we can solve

$$\hat{s}=\frac{\left(1-\tau\right)w-\left(\beta R\right)^{-\frac{1}{\gamma}}\left(w\hat{l}+\tau w\right)}{\left(\beta R\right)^{-\frac{1}{\gamma}}R+1}$$

So

$$k = \frac{\hat{s}}{1+\hat{l}} = \frac{(1-\tau)w - (\beta R)^{-\frac{1}{\gamma}}\left(w\hat{l} + \tau w\right)}{\left((\beta R)^{-\frac{1}{\gamma}} R + 1\right)\left(1+\hat{l}\right)}$$

Also, note that in the steady state, the resource constraint implies that

$$\begin{aligned}
&c_1 + c_2 + K = F(K, L) = RK + wL \\
&c_1 + c_2 = (R-1)K + wL \\
&(\beta R)^{-\frac{1}{\gamma}} c_2 + c_2 = (R-1)(1+l)k + w(1+l) \\
&\frac{1+(\beta R)^{-\frac{1}{\gamma}}}{1+l} = \frac{(R-1)k + w}{c_2}
\end{aligned}$$

Plug in $k = \frac{(1-\tau)w - (\beta R)^{-\frac{1}{\gamma}}\left(w\hat{l}+\tau w\right)}{\left((\beta R)^{-\frac{1}{\gamma}} R+1\right)\left(1+\hat{l}\right)}$ and $\frac{(R-1)k+w}{c_2} = \frac{1+(\beta R)^{-\frac{1}{\gamma}}}{1+l}$, we get

$$\begin{aligned}
&A - B \\
&= \frac{1}{1-\hat{l}}\left[1 - (\beta R)^{\frac{1}{\gamma}} + 2\frac{(\beta R)^{\frac{1}{\gamma}} + R}{w}k - \left(1-\hat{l}\right)\frac{w + (R-1)k}{c_2}(\beta R)^{\frac{1}{\gamma}}\left(1 - \frac{2\tau}{1-\hat{l}}\right)\right] \\
&= \frac{1}{1-\hat{l}}\left[1 - (\beta R)^{\frac{1}{\gamma}} + 2\frac{(\beta R)^{\frac{1}{\gamma}} + R}{w}\frac{(1-\tau)w - (\beta R)^{-\frac{1}{\gamma}}\left(w\hat{l}+\tau w\right)}{\left((\beta R)^{-\frac{1}{\gamma}} R+1\right)\left(1+\hat{l}\right)} - \left(1-\hat{l}\right)\frac{1+(\beta R)^{-\frac{1}{\gamma}}}{1+l}(\beta R)^{\frac{1}{\gamma}}\left(1 - \frac{2\tau}{1-\hat{l}}\right)\right] \\
&= \frac{1}{1-\hat{l}}\left[1 - (\beta R)^{\frac{1}{\gamma}} + 2(\beta R)^{\frac{1}{\gamma}}\frac{1-\tau-(\beta R)^{-\frac{1}{\gamma}}\left(\hat{l}+\tau\right)}{1+\hat{l}} - \frac{(\beta R)^{\frac{1}{\gamma}}+1}{1+\hat{l}}\left(1-\hat{l}-2\tau\right)\right] \\
&= 0
\end{aligned}$$

So $\hat{s}_\tau - k\hat{l}_\tau\Big|_{\text{steady- state}} = \frac{\gamma\beta R\left(wv'\right)^2}{c_2}\frac{B}{D_2}(1-\gamma)$, which implies $sign\left(\frac{dk}{d\tau}\right) = sign(1-\gamma)$ for all τ.

CHAPTER 2

PARETO-IMPROVING ENVIRONMENTAL POLICIES UNDER DEBT SUSTAINABILITY

2.1 Introduction

For centuries, most of mankind struggled with fundamental, existential issues such as food, shelter, survival; concerns about global warming and climate change hardly ever occupied anyone's attention. There was really no need for such environmental concerns anyway. Climatologists tell us that, before 1750, atmospheric concentrations of CO_2, CH_4, and N_2O -- three important long-lived greenhouse gases -- were quite low.[7] The Industrial Revolution changed all that. Economic growth, stuck for a millennium at around 0.5% per year, doubled to 1% or higher in Western Europe and later in the Americas. Along with increased prosperity came deforestation (to make way for agriculture) and the burning of fossil fuels (to supply the energy needs of growing economies). Atmospheric concentration of CO_2 started increasing; current levels are 35% higher than what they were in 1850. The U.S. Department of Energy estimates, if nothing is done, CO_2 concentrations could reach 900 ppm by the year 2100. The IPCC reports that such increases in concentrations of greenhouse gases could cause global temperatures to rise substantially, even 4-6 $^{\circ}C$ or higher. Not a day goes by when climate scientists, economists, even politicians don't raise alarm over the expected dire consequences of such temperature rises: the list

[7] Before 1750, concentrations of CO_2 and CH_4 were below 280 ppm and 790 ppb, respectively. These days, concentrations of CO_2 are about 390 ppm and CH_4 levels exceed 1,770 ppb. Climatologists document that these numbers are much higher than at any time during the last 650,000 years. Climate models suggest that only 5% of the total extra CO_2 in the atmosphere – the CO_2 that wouldn't be there if humans didn't create it -- date back to pre-1850.

is long and includes substantial sea-level rises, sharp decline in agricultural output, coastal erosion, demise of fish stocks, rise in the frequency of severe weather events, and massive ecosystem upheavals. Today climate change and global warming are viewed by many as top-level existential threats, much in the same way food, shelter, and survival concerns occupied the minds of our distant ancestors.

Unlike these other existential concerns, climate change is an externality -- the Stern Review (2007) calls it the greatest market failure and largest externality in history. What is more, both the externality and the measures needed to address it are, necessarily, intergenerational in nature. After all, greenhouse gases are long-lived and their effects linger, long after they appear. Similarly, costly measures adopted today may generate benefits far into the future, well beyond the lifespan of the generations funding them. As such, any response towards combating climate change will require strong political action across generations. By the same token, however, any policy response will likely create intergenerational winners and losers and, in turn, raise thorny questions about intergenerational equity and its trade-off with efficiency. Pearson (2011) phrases it bluntly: Should we sacrifice our use of cheap fossil fuel energy today so that generations yet unborn, who presumably will be richer than we are, can avoid adjusting to a warmer world? Do we owe the future a clean planet? Even if we could agree that the answer to the previous question is yes, how should the near-term costs of clean-up be allocated across generations in a fair and efficient manner? For if it is not perceived to be fair, why would different generations participate in this cross-generational initiative? There are other concerns of a more practical nature. Specifically, there is no political institution or mechanism through which the present generation can securely compensate [...future] generations for the consequences of global warming [...and symmetrically,] there is no obvious way for future generations [...] to compensate us for our sacrifices if we take expensive greenhouse abatement measures today. (Pearson, 2011; pp. 23) Is it possible to navigate our way around these concerns?

The by-now vast literature on Integrated Assessment Models (IAMs) has taken up aspects of these overarching themes by uniting the science and economics of climate change via the damage function, a way of identifying the impacts of climate change and attributing monetary damages to them. Their goal, in many instances, is to maximize a global welfare criterion within the confines of an infinitely-lived-agent, Ramsey-style model and the control variable is an abatement policy. The point is to solve for optimal paths for consumption, investment, and other variables while devoting enough resources to keeping the environment clean. Intergenerational conflict of the type discussed above is a mere sideshow in the one-agent, one-world scenario that IAMs study.

In this paper, we are conceptually motivated by the same sort of big questions that occupy the IAMs but our goal is far more modest: it is to write down a simple, analytically tractable (and therefore, stylized) model that captures the essence of the intergenerational conflict discussed above.[8] Specifically, we study a small open, overlapping-generations economy in which laissez faire -- business as usual (BAU) -- is associated with declining consumption, declining welfare and worsening of the environment over generations. A government contemplates policy action aimed at correcting the underlying intergenerational externality by cleaning up the environment. Allowing for the fact that heat-trapping gases such as carbon dioxide stay on for generations, benefits from the government's clean-up effort would take time to ramp up. Understandably, such a policy would create winners and losers, across cohorts. The standard response to these inter-

[8] Our goal is not to produce a reasonably accurate model of climate-change economics nor is it to study specific policies suggested as ways to combat climate change.

generational equity concerns would be to postulate a social welfare function, a weighted function of the welfare of current and future generations, the likely losers and winners respectively. Depending on the weights chosen -- a matter of considerable importance as discussed in the Stern Review (2007) -- policy action may be justified if it maximizes the generationally-aggregated social welfare function even if it requires some generations to sacrifice for others. We refocus the issue. Along with several recent papers -- Bovenberg and Heijdra (2002), Hoel et al. (2015), von Below et al. (2015), and others -- we sidestep the contentious issue of assigning weights to generations and directly ask, can policy action be rationalized even after imposing the Pareto criterion, the restriction that no generation subsequent to policy action be made worse off than if business as usual had continued?

Why might we think such a line of questioning may yield answers? Standard welfare economics, recently emphasized by Foley (2008) and Heal (2009), suggests that in the presence of a huge uninternalized externality such as climate change, the business-as-usual scenario cannot be Pareto efficient and hence action to correct the externality must, in principle, offer a Pareto improvement: the gains must outweigh the costs so that the gainers could compensate the losers and still gain. We can all come out ahead --- whether we actually do is a matter of institutional design. (Heal, 2009) This remark from Heal (2009) captures the essence of our endeavor. To begin with, we employ insights from overlapping-generation models of deficit financing -- see Bovenberg and Heijdra (1998) -- to address Pearson's (2011) concern about there being no obvious way by which generations can share clean-up costs: we allow for inter-generational compensation for investments in environment-friendly policies via debt financing.[9] By insisting that such policies

[9] As Sachs (2015) puts it, “[t]his is an option too rarely discussed in the current debate”.

meet the generational Pareto criterion, we are in effect arguing that Pearson's other concern -- lack of a political institution -- is not that critical: after all, it seems natural to think that policies that satisfy the Pareto criterion are less likely to be blocked as they make their way through modern democratic processes. What makes our analysis especially challenging is the fact that the very act of compensating current generations releases its own dynamics. For sure, investments in environment-friendly policies via debt financing allow future generations to reap gains, but they also have to participate, via tax payments and additional debt purchase, in the servicing of the outstanding debt. Debt will be growing at the gross rate of interest (assumed to exceed unity) which means it is by no means trivial whether the downstream gains from a better environment can cover the aforediscussed compensation (including interest) and prevent the debt from exploding. There is the added complication that the tax instrument we study is distortionary: it affects incentives to produce, with feedback effects on both the budget and the environment and other variables influencing welfare. In sum, one major contribution of this paper lies in the demonstration of the possibility that Pareto improvements over BAU are possible and that the associated path of debt does not misbehave.

There is another important dimension in which we advance the literature. As Karp and Rezai (2014a) argues, a convergent conclusion from the literature emanating from the Stern Review (2007) and the IAMs -- summarized in Heal (2009) -- is that current generations must sacrifice consumption in order to combat climate change.[10] This conclusion is often blamed as the

[10] Kolbert (2008) profiles the goals and aspirations of a Swiss organization, the 2,000-Watt Society, which pushes for human beings to lead a life using less than two-thousand-watts of energy in a day. Evidently, the average Swiss uses about 5,000 watts and bringing this down to 2,000 would require a significant reduction in every realm: as Kolbert (2008) puts it, a person who continued to travel and use electricity at current rates would consume two thousand watts without having anywhere to live or work, or anything to eat. Nordhaus (2007) argues that assumptions in the Stern Review (2007) concerning a low discount factor amplifies the harmful impacts of climate change in the distant future and rationalizes deep cuts in emissions, and indeed in all consumption, today. Rezai

reason why climate negotiations between countries have proven to be a non-starter.[11] We take on the challenge of studying policies that not only satisfy the generational Pareto criterion in utility terms, but also ensure that no generation has to sacrifice consumption along the way.[12] Our results connect up with the broader literature on sustainability -- Neumayer (2007), Heal (2009) -- that recognizes substitutability between a loss to environmental capital (due to global warming) and gains to incomes/ consumption and argues for the need to maintain at least a minimum critical level of the former. In a way, requiring that consumption not decline ties our hands substantially; it precludes the possibility of exploiting the substitutability of the environment and consumption to leave generations as happy as in the BAU.

The literature on the economics of inter-generational equity and efficiency concerns in environmental models is vast. Below, we summarize some of the papers that are closest in spirit to the current endeavor. An early contribution that led the way in terms of the search for Pareto-improving policies is Gerlagh and Keyzer (2001) study a productive, non-renewable natural resource with amenity value and show that handing over property rights over that resource to an intergenerational trust fund that entitles every generation to the same income claim as in the zero extraction policy can yield a Pareto improvement.[13] Of course, for the initial generation to want to

(2010) argues this need to cut consumption is an artifact of the constraining assumption made in IAMs that, in spite of knowing about the dangers of climate change, agents in the BAU, invest nothing in mitigation, a constrained-optimal equilibrium. This automatically implies current generations would attain lower consumption and utility levels if they started investing in mitigation.

[11] The American way of life is not up for negotiation -- the classic U.S. position outlined by the senior President George Bush at 1992 Earth Summit in Rio de Janeiro.

[12] That something like this may be possible has been discussed, informally, in Foley (2008).

[13] Rasmussen (2003) is an early example of a paper using a calibrated OG model to study environmental taxation in a model where environmental quality is held fixed. Leach (2009) is similar in spirit but stays away studying tax or debt policies.

create the fund requires them to care about future generations. An important contribution involving intergenerational borrowing is Bovenberg and Heijdra (1998). In their setup, distant and near generations differ in their reliance on capital income (which translates to non-environmental welfare) versus environmental utility. Taxes on pollution are akin to a tax on capital and benefit distant generations -- they enjoy a better environment -- but hurt them because they inherit a smaller stock of physical capital. Debt, as in our paper, can be used to allow all generations to share in the efficiency gains of environmental policy, in some cases in a Pareto-improving way.[14] The assumption that pollution hurts utility directly but does not affect production makes our results non-comparable; additionally, only marginal policy changes relative to the BAU are considered which means they can sidestep issues relating to long-run behavior of debt paths.

Karp and Rezai (2014a, b) focus on the conflict between different types of agents alive when the [mitigation] policy is first implemented. They depart from the usual one-sector OG model and allow for two sectors with an endogenously-evolving relative price between the sectors. They rely on an idea, reminiscent of Poutvaara (2003), that if investments in pollution mitigation by the current young generate increases in future asset values, then the current old -- the owners of said assets -- can compensate the young from those capitalized benefits leaving everyone better off, just as Heal (2009) argued would be possible. Conceptually, the novelty of their paper rests on the fact that tax-induced increases in asset prices allow market-intermediated, Pareto-improving

[14] Our work is part of an important literature that studies the consequences of environmental policies on environmental quality, growth and welfare (Howarth and Norgaard 1992; John and Pecchenino 1994; Jouvet et al. 2000; Gutierrez 2008; Goenka et al. 2012; Dao and Davila, 2014; Wang et al, 2015). In many of these papers, environmental quality enters preferences directly. Their primary purpose is to study the role of government for eliminating the dynamic inefficiency in OG economies with environmental externalities. These papers focus on tax-financed mitigation policies and do not allow for debt financing. Intergenerational equity concerns or the search for Pareto-improving policies is not their focus. Fodha and Seegmuller (2014) do allow for debt financing but stay away from studying welfare issues along the transition.

policies even when the government cannot use bonds to redistribute across generations. Dao et al. (2015) study an intergenerational social compact between generations in which the young invest some of their labor income in mitigation activities in return for a subsidy to their old-age capital income paid for by the next young generation. The compact terms are such that participation in it generates a Pareto improvement compared to non-participation. They also consider compact terms that are self-sustaining, meaning any incentive to default on the contractual terms are eliminated. In an important recent contribution, von Below et. al (2015) revisit the Poutvaara (2003) and Karp and Rezai (2014a, b) strategy of resolving the conflict between the young and the old at the point the mitigation policy is implemented. In their setup, the old and young suffer losses in rental and wage income when energy use (which is polluting) is curtailed but the old can be offered a compensatory pension by the young in lieu of the future benefit the latter get from a better environment. The added novelty comes from the fact that the benefit from a cleaner environment accrues not just to the current young in the future, but to all future generations; if the future beneficiaries can be co-opted into the deal between the current young and old, then far more ambitious environmental policies can be attempted without hurting any generation. Their focus, however, is not on the behavior of the path of pensions nor are they seeking policies that always improve upon consumption in the BAU.

The rest of the paper is organized as follows. Section 2.2 describes the model economy and the business as usual environment, and exposes the inefficiency. Section 2.3 studies an environmental policy under a generational Pareto criterion. Section 2.4 presents a tractable case with quasi-linear utility and Section 2.5 studies the robustness of the results. Finally, Section 2.6 concludes. Proofs of results are contained in the appendices.

2.2 The Model Economy: BAU

2.2.1 Preliminaries

We consider a one-good, small, open economy inhabited by two period-lived generations of agents and an infinitely-lived policymaker. At each date $t+j=0,1,2,...$ a unit mass of identical agents is born. The laissez faire economy is called "business as usual" (BAU). Agents are called "young" ("old") in the first (second) period of life. Young agents are endowed with one unit of labor and nothing when old. The young also have access to a production technology (F) that uses labor (L) as the only input to produce the single, consumption good. We assume $F(L)$ has standard properties, $F_L(L)>0, F_{LL}(L)\leq 0$ with the implication that $\frac{F(L)}{L}\geq F_L$. All agents have complete access to perfect international capital markets and can save at a fixed, gross return R.

Pollution is generated as a by-product of productive activities. Let total pollutant emissions from productive activities in period $t+j$ be denoted by e_{t+j} and let the stock of pollutants at the start of $t+j$ be denoted by S_{t+j}. Then, the stock of pollution at the start of $t+j+1$ is described by

$$S_{t+j+1}=(1-\varepsilon)S_{t+j}+e_{t+j}, S_0>0 \text{ given} \tag{2.1}$$

Where $\varepsilon\in(0,1)$ is a constant that determines the speed with which pollution levels return to zero in the absence of any fresh emissions. Notice how eq. (1) captures the idea that changes to the environment can be very long-lived, spread across many cohorts. Since labor is the only input, we think of it as the polluting input as well: we posit that use of input L_{t+j} generates emissions of amount $e_{t+j} = G\left(L_{t+j}\right)$ where $G(0)=0$ and $G_L(\cdot)>0$.[15] Emissions can increase more (less) rapidly than input use if G is assumed to be convex (concave) -- see Heutel (2012). Pollution is an unintended by-product of productive activity by firms and no firm-level disposal of this by-product is possible. (Murty et al., 2012)

As is standard, pollution reduces the production of output as captured by a damage function, $H(S)$, where $H(0)=1, H_S(S)<0$ for $S>0$. Some of the analytics in the study of debt dynamics below will be conducted with a linear approximation

$$H(S) = 1-\rho S; \rho > 0 \text{ and } \rho \approx 0 \tag{2.2}$$

which satisfies $H(0)=1$ and $H_S(S)=-\rho<0$. Net output (denoted by y in period $t+j$ is given by

$$y(S_{t+j}, L_{t+j}) = H\left(S_{t+j}\right)F\left(L_{t+j}\right),$$

where it is seen that $y_S<0, y_L>0$ and $y_{SL}<0$. Pollution reduces net output (since $H_S(S)<0$) that is available for consumption.[16] Also, while the marginal product of labor is positive, it declines with pollution: $H(S)$ acts as an adverse productivity shock. Young agents produce at $t+j$ taking S_{t+j} as given and $H\left(S_{t+j}\right)$ captures the damage to their output caused by this level of pollution. The point is, S_{t+j} was created by the productive actions of their ancestors who did not internalize

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[15] Our results are unaffected if G depends on gross output. We allow for G to depend on net output in the numerical section.

[16] The Stern Review (2007) uses 5% of GDP as the lower bound for the cost of climate change under the BAU scenario. Burke et al. (2015) show that overall economic productivity is nonlinear in temperature for all countries, rich or poor, with productivity falling sharply at temperatures higher than 13 ^{o}C, and that the relationship is true for both agricultural and non-agricultural activity. Dell et al. (2012) finds strong growth effects but only for poor countries.

the downstream damage; after all, agents are not altruistic and changes in the environment have no direct effect on them.

We assume agents born in period $t+j$ care only about young-age leisure $\left(l_{t+j} \equiv 1-L_{t+j}\right)$, and old-age consumption $\left(c_{t+j+1}\right)$. The utility function for generation $t+j$ is given by $U_{t+j} \equiv u\left(c_{t+j+1}\right)+v\left(l_{t+j}\right)$ where $u(\cdot)$ and $v(\cdot)$ satisfies standard properties, $u_c > 0 \geq u_{cc}$ and $v_l > 0 \geq v_{ll}$. In much of what we do below, we assume an iso-elastic form for u:

$$u\left(c_{t+j+1}\right)=\frac{\left(c_{t+j+1}\right)^{1-\sigma}}{1-\sigma};\sigma \geq 0. \tag{2.3}$$

It is apparent that the relevant Arrow-Pratt measure is $\frac{-cU_{cc}}{U_c}=\sigma$. For future use, the special case of $\sigma=0$ will be referred to as quasi-linear utility. Also notice S (or more generally, environmental quality) does not enter agents' utility directly. Importantly, there is no altruism on the part of agents.[17]

2.2.2 Discussion of modeling assumptions

The model setup is necessarily barebones, designed to generate clean qualitative insights taking advantage of a lot of analytical tractability. Unlike John and Pecennino (1994), we do not embed environmental quality in the utility function, which allows us to sidestep issues relating to the substitutability between environmental and consumption goods, for as Neumeyer (2007) and others have argued, if the substitutability is low, then it may be that no consumption growth,

[17] We disallow altruism on the part of agents not because we think people don't care about the welfare of their progeny but because we wish to make a case for environmental action even if they didn't. Allowing altruism would also introduce private transfers from parents to children some of which may be crowded out by the government's debt policy.

however high, can compensate for the damage to the environment -- after all, as Heal (2009) points out, certain ecosystem services or products, such as water and food, are essential to survival and cannot be replaced by produced goods.

The assumption of a single input, labor, is obviously limiting but also keeps the analysis manageable. Studying a closed economy along with capital as an additional input, possibly the dirty input, would add another state variable, bring in interest rate effects, and clutter the dynamics considerably. As will become clear, our focus is largely on the issue of implementation of environmental policies under a generational Pareto criterion. Adding capital does not fundamentally alter our understanding of that issue.

Unlike Integrated Assessment Models or models studying climate change more generally, we make no attempt to connect pollution with climate change and global temperature rises (with associated output losses for agriculture, sea-level rises, ecosystem disruptions, damage to fish stocks, etc.) Nor do we think in terms of a single world economy. Concrete examples of what we have in mind include water pollution that is limited to a country, localized atmospheric pollution over a region, and so on.

2.2.3 Agent's problem

Agents are atomistic producers who use their production technology to produce output of amount $H\left(S_{t+j}\right)F\left(L_{t+j}\right)$. The primary decision of the agent is his choice of labor input. For future use, we introduce a tax τ_{t+j} on labor. (The tax and its use is discussed in Section 2.3 below. For now, simply treat τ as a parameter.) Since agents do not consume when young, they save their entire after-tax, young-age income, $\left[H\left(S_{t+j}\right)F\left(L_{t+j}\right)-\tau_{t+j}L_{t+j}\right]$, at the world interest rate, $R \geq 1$.

They take R, S_{t+j} and τ_{t+j} as given, and choose L_{t+j} to maximize their utility subject to the budget constraint $c_{t+j+1} = R\left[H\left(S_{t+j}\right)F\left(L_{t+j}\right) - \tau_{t+j}L_{t+j}\right]$. The first order condition (assuming an interior solution[18]) is given by

$$U_L = u_c\left(\cdot\right)R\left[H\left(S_{t+j}\right)F_L\left(L_{t+j}\right) - \tau_{t+j}\right] - v_l\left(1 - L_{t+j}\right) = 0. \tag{2.4}$$

It is easy to check that $U_{LL} < 0$, implying the second order condition is satisfied. Eq. (2.4) implicitly defines an optimal labor supply function, $L_{t+j} = L\left(S_{t+j}, \tau_{t+j}\right)$. In the BAU, the corresponding labor supply function is given by $L_{t+j}^{BAU} = L\left(S_{t+j}^{BAU}\right)$.

In this general form, lifetime indirect utility $\tilde{U}$ equals

$$\tilde{U}_{t+j}\left(S_{t+j}, \tau_{t+j}\right) = u\left(\underbrace{R\left[H\left(S_{t+j}\right)F\left(L_{t+j}\left(S_{t+j}, \tau_{t+j}\right)\right) - \tau_{t+j}L_{t+j}\left(S_{t+j}, \tau_{t+j}\right)\right]}_{c_{t+j+1}}\right) + v\left(1 - L_{t+j}\left(S_{t+j}, \tau_{t+j}\right)\right) \tag{2.5}$$

and under BAU, it is given by

$$\tilde{U}_{t+j}^{BAU}\left(S_{t+j}^{BAU}\right) \equiv u\left(\underbrace{R\left[H\left(S_{t+j}^{BAU}\right)F\left(L\left(S_{t+j}^{BAU}\right)\right)\right]}_{c_{t+j+1}^{BAU}}\right) + v\left(1 - L\left(S_{t+j}^{BAU}\right)\right). \tag{2.6}$$

We start by asking, how do agents' labor supply respond if the stock of pollution is higher?

Lemma 2.1 *The sign of the comparative static labor supply responses,* $\frac{\partial L_{t+j}}{\partial S_{t+j}}$ *and* $\frac{\partial L_{t+j}}{\partial \tau_{t+j}}$ *are, in general, ambiguous. In the BAU, we have*

$$\text{sign } \frac{dL\left(S_{t+j}^{BAU}\right)}{dS_{t+j}^{BAU}} = \text{sign } \left(\frac{-c_{t+j+1}^{BAU}u_{cc}\left(\cdot\right)}{u_c\left(\cdot\right)} - 1\right) = \text{sign } \left(\sigma - 1\right) \text{ for (2.3).} \tag{2.7}$$

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[18] If we assume $\lim_{c\to 0} u_c\left(c\right) = +\infty$ and $\lim_{l\to 0} v_l\left(l\right) = +\infty$, then we must have $L_{t+j} \in \left(0,1\right)$.

The expressions for $\frac{\partial L_{t+j}}{\partial S_{t+j}}$ and $\frac{\partial L_{t+j}}{\partial \tau_{t+j}}$ outside of the BAU as well as the proof of Lemma 2.1 are in Appendix 2.A. In the BAU, an increase in the stock of pollution reduces the marginal product of labor (which acts as a tax on labor supply) causing agents to want to substitute into leisure. However, there is a countervailing effect: as the environment worsens, net output (income) decreases inducing a wealth effect and incentivizing agents to want to supply more labor. The net effect is ambiguous and, as in the textbook models of labor supply, the net effect depends on risk aversion parameters. What happens to consumption?

Lemma 2.2 *(a)* $\frac{\partial c_{t+j+1}}{\partial S_{t+j}} < 0$, *(b)* $\frac{\partial c_{t+j+1}}{\partial \tau_{t+j}} < 0$.

As the environment worsens, net output (income) decreases, shrinking the consumption possibilities of the agent. What happens to the welfare of agents?

Lemma 2.3 *(a)* $\frac{\partial \tilde{U}_{t+j}\left(S_{t+j}, \tau_{t+j}\right)}{\partial S_{t+j}} < 0$, *(b)* $\frac{\partial \tilde{U}_{t+j}\left(S_{t+j}, \tau_{t+j}\right)}{\partial \tau_{t+j}} < 0$.

Agents' utility depends on their consumption and leisure. The effect of pollution on their labor supply (leisure) is washed out by the envelope theorem. What remains is the indirect effect of pollution on the output they produce -- via the damage function -- which goes in a negative direction. A worsening environment hurts the welfare of every generation. The effect of a higher tax on the labor input -- the income, substitution effects -- are washed out by the envelope theorem; what remains is the direct effect on after-tax income which reduces consumption.

2.2.4 Dynamics and steady state: BAU

We close our discussion by asking, how does environmental quality evolve along the transition therein? Using eq. (2.1), it follows that the dynamics of pollution in the BAU is described by the first-order, (possibly) non-linear difference equation

$$S_{t+j+1}^{BAU} = (1-\varepsilon) S_{t+j}^{BAU} + G\left(L\left(S_{t+j}^{BAU}\right)\right) \tag{2.8}$$

where, recall $\varepsilon \in (0,1)$ and $G_L(\cdot) > 0$. A non-trivial steady state in the BAU, denoted by S_{BAU}^*, is a fixed point of

$$S_{BAU}^* = \frac{G\left(L_{BAU}^*\right)}{\varepsilon}; L_{BAU}^* \equiv L\left(S_{BAU}^*\right) \tag{2.9}$$

Existence and uniqueness issues are dealt with using standard techniques.[19] Similarly, imposing parametric restrictions such that

$$0 < \left.\frac{dS_{t+j+1}^{BAU}}{dS_{t+j}^{BAU}}\right|_{S_{BAU}^*} < 1 \Leftrightarrow 0 < 1 - \varepsilon + G_L\left(L\left(S_{BAU}^*\right)\right) \left.\frac{dL\left(S_{t+j}^{BAU}\right)}{dS_{t+j}^{BAU}}\right|_{S_{BAU}^*} < 1 \tag{2.10}$$

holds ensures local stability of S_{BAU}^*. If, in addition, we assume $S_0 < S_{BAU}^*$, then it is apparent that the stock of pollution in the BAU is increasing and converging to S_{BAU}^* in the long run.

2.2.5 Inefficiency

Intuitively, agents do not take into account the effect of their production decisions on the pollution that generates which, in turn, affects the production decisions of their offspring. To see

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[19] Define $J(S) \equiv G(L(S)) - \varepsilon S$. Then, it follows $J(0) = G(L(0)) > 0$ and $\lim_{S\to\infty} J(S) = -\infty$ if G has a finite upper bound. If $\sigma \le 1$, then $L_S \le 0$, so $J_S(S) = G_L(\cdot) L_S(S) - \varepsilon < 0$, in which case S_{BAU}^* is unique.

this more formally, write out the agent's problem in steady state. An agent takes S as given to solve

$$\max_L U^P \equiv u(c)+v(1-L)$$
$$\text{s.t. } c = RH(S)F(L)$$

The first order condition is $u_c(c)RH(S)F_L(L)-v_l(1-L)=0$ which, combined with $\varepsilon S = G(L)$ -- see (eq. 2.9) yields

$$u_c\left(RH\left(\frac{G(L^P)}{\varepsilon}\right)F(L^P)\right)RH\left(\frac{G(L^P)}{\varepsilon}\right)F_L(L^P)-v_l(1-L^P)\equiv 0$$

where L^P is the solution to the individual's problem.[20] Now consider the problem of a social planner who solves

$$\max_L U^{SP} \equiv u(c)+v(1-L)$$
$$\text{s.t. } c = RH(S)F(L), \varepsilon S = G(L)$$

incorporating the effect of labor supply on the environment. Denote the planner's solution by L^{SP}. Then,

$$u_c\left(RH(\cdot)F(L^{SP})\right)R\left[\underbrace{H_S(.)\frac{G_L(L^{SP})}{\varepsilon}F(L^{SP})}+H\left(\frac{G(L^{SP})}{\varepsilon}\right)F_L(L^{SP})\right]-v_l(1-L^{SP})=0.$$

For L^{SP} to be well-defined, assume $\frac{d^2U^{SP}}{dL^2}<0$ holds. (Even though $\frac{d^2U^P}{dL^2}<0$ obtains, it does not follow that $\frac{d^2U^{SP}}{dL^2}<0$ holds, precisely because of the externality.) Notice, the underscored term arises because the planner takes into account the effect of his choice of L on S which, in turn, affects $H(S)$.

[20] As discussed earlier, $\frac{\partial^2U^P}{\partial L^2}<0$ holds and so L^P is well defined.

Proposition 2.1 *In the steady state, the private agent oversupplies labor (which translates into a worse environment) relative to the social planner, i.e.,* $L^P > L^{SP} \Leftrightarrow S^P > S^{SP}$.

2.2.6 A portrait of the BAU

In the regime without any sort of governmental involvement, the BAU, polluting emissions are a by-product of production using the (dirty) input, labor. Starting from a low initial level of pollution, S_0, under the conditions spelt out in (2.10), the stock of pollution continues to rise and approaches a higher level, S^*_{BAU}, in the long run. This implies a steady, unrelenting decline in the quality of the environment. Along such a path of environmental degradation, it follows from Lemma 2.2 that consumption declines as well. In some utility specifications, labor supply rises along this transition, which, together with the consumption decline, serves to hurt every generation (See Figure 2.1).

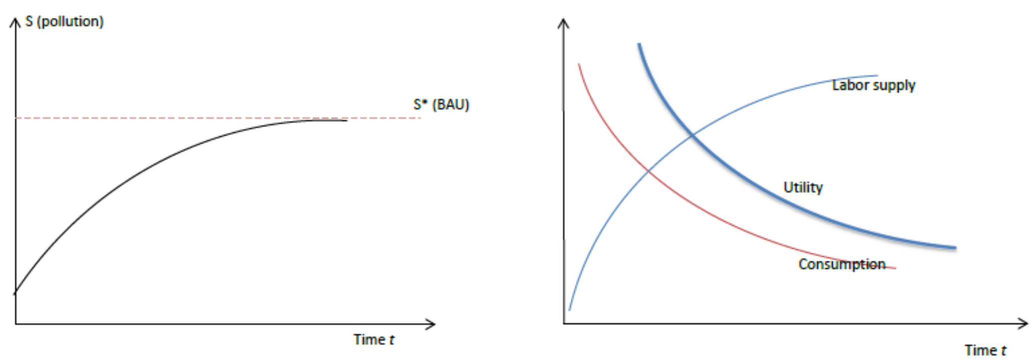

Figure 2.1. A portrait of the BAU, $\sigma > 1$

Every generation pollutes and leaves a worse environment for its progeny than what it received from its parents. The worsening environmental quality hurts the children but parents -- the people whose actions generated the pollution -- do not care (since they are not altruistic). The big question for us is, can the government initiate a pollution-reduction policy that leaves every generation, post policy, no worse off, possibly better off, compared to their life in the BAU?

2.3 Environmental Policy

The government wishes to implement a pollution-abatement policy designed to reduce pollution and improve environmental quality. We posit that the government commits to abating a fraction μ_{t+j} of the total emissions in period $t+j$, at a cost $A(\mu_{t+j};e_{t+j})$ where

$$A(\mu_{t+j};e_{t+j}) = \Lambda\left(\mu_{t+j}\right)e_{t+j} = \Lambda\left(\mu_{t+j}\right)G\left(L_{t+j}\right). \tag{2.11}$$

We assume $\Lambda\left(0\right)=\Lambda_{\mu}\left(0\right)=0, \Lambda_{\mu}>0, \Lambda_{\mu\mu}>0$ for $\mu>0$: the abatement cost is 0 and minimum under BAU, and it is increasing and convex for positive levels of abatement. Under such a policy, the after-abatement total emissions in period $t+j$ is $e_{t+j}=\left(1-\mu_{t+j}\right)G\left(L^{\mu}_{t+j}\right)$ and the transition equation for the stock of pollution is given by

$$S^{\mu}_{t+j+1}=\left(1-\varepsilon\right)S^{\mu}_{t+j}+\left(1-\mu_{t+j}\right)G\left(L^{\mu}_{t+j}\right) \tag{2.12}$$

where the superscript μ reminds us that the variable in question is in the policy regime.

2.3.1 Government finances and debt policy

In our world, the government is entrusted with the task of reducing the level of pollution in the economy. The government's pollution abatement efforts may start at any date, taking as given the level of pollution under BAU, S_t^{BAU}. At any date, there are two modes of financing abatement expenses, either by taxation $\left(\tau_{t+j}\right)$ or by public borrowing on the international capital market at rate R (with associated debt and tax dynamics which we return to below).

A few definitions and notation descriptions are in order. Let B_{t+j} denote the stock of one-period government debt at the end of period $t+j$; assume $B_{t-1}=0$ (that is, there is no outstanding debt in the BAU). Also, let M_{t+j} denote the primary (i.e., non-interest) budget balance in $t+j$, the difference between tax revenue and primary expenditure (abatement cost). Then,

$$M_{t+j} = \tau_{t+j} L^{\mu}_{t+j} - \Lambda\left(\mu_{t+j}\right) G\left(L^{\mu}_{t+j}\right) \tag{2.13}$$

where L^{μ}_{t+j} satisfies (2.4). Finally, define T_{t+j} as the total budget balance, that is the primary balance where interest payments to foreign lenders are added on to abatement expenses. Then,

$$T_{t+j} = \tau_{t+j} L^{\mu}_{t+j} - \left[\Lambda\left(\mu_{t+j}\right) G\left(L^{\mu}_{t+j}\right) + \left(R-1\right) B_{t+j-1}\right] = M_{t+j} - \left(R-1\right) B_{t+j-1}. \tag{2.14}$$

If $T_{t+j} > 0$, we say there is a government budget surplus; a negative balance is called a government budget deficit. Debt service is defined as the sum of principal repayments, B_{t+j-1} (because these are one-period bonds) and interest payments, $\left(R-1\right) B_{t+j-1}$. Fresh debt at $t+j$, must cover debt service on previous debt and any primary budget shortfalls, i.e., $B_{t+j} = RB_{t+j-1} - M_{t+j}$ from where it follows that change in the stock of bonds satisfies $B_{t+j} - B_{t+j-1} = -T_{t+j}$.

Using the definitions above, write

$$B_{t+j+1} - B_{t+j} = R\left(B_{t+j} - B_{t+j-1}\right) - \left(M_{t+j+1} - M_{t+j}\right) \tag{2.15}$$

Suppose $M_{t+j+1} - M_{t+j} < 0$, i.e., the primary budget balance deteriorates over time. Then, it is apparent that $B_{t+j+1} - B_{t+j} > 0$ meaning debt levels increase at each date. Since $R > 1$, debt levels along such a trajectory would explode rendering the policy fiscally unsustainable. Therefore, it becomes imperative the government can raise enough in tax revenue to prevent this from happening. A necessary condition for debt paths to be sustainable is that M turns positive at some date and $M_{t+j+1} - M_{t+j} > 0$ at some other date as well. For future use, define the debt turning point as the first date, say $t+k$, for which $B_{t+j} < B_{t+j-1}$ holds, that is the first date for which the current debt level is below its immediate predecessor: $B_{t+k} < B_{t+k-1}$. It follows that $T_{t+k} > 0$, there is a government budget surplus in $t+k$. The relevance of the debt turning point is this. Suppose a debt turning point is reached at $t+k$ implying the first term on the r.h.s of eq. (2.15) is negative: $\left(B_{t+k} - B_{t+k-1}\right) < 0$. If at that date, M is positive, and $M_{t+j+1} > M_{t+j}, \forall j > k$, the second term on the r.h.s of eq. (2.15) is also negative and together with the negative first term would imply $B_{t+j+1} < B_{t+j}, \forall j > k$. In other words, if M is positive and rising over time (i.e., the gap between tax revenue and abatement expenses is positive and rising), then, debt levels once they fall -- the debt turning point is reached -- continue to fall forever after. Indeed, since it is assumed $R > 1$, the debt level will reach zero in finite time. At that point, the country is debt free; if it so wishes, it can continue to raise taxes to pay for pollution abatement and any primary surplus may be lent to world markets or rebated to taxpayers.

2.3.2 Implementation hurdles and the Pareto criterion

It seems possible that government intervention via pollution abatement may generate welfare gains downstream relative to life under the BAU. After all, lower pollution levels in the future will, ceteris paribus, boost output and consumption. But these welfare improvements may take a while to appear since pollution abatement is a slow process. Does this mean some initial generations will have to sacrifice for the benefit of the future ones? Could it be possible for society to reap the distant gains of government intervention all the while ensuring no generation is harmed along the way?

Answers to these questions are not trivial. The inaugural generation does not benefit from the pollution reduction -- at the point the policy is introduced, the pollution level and hence their labor supply is predetermined at BAU levels, and therefore the government must incur a debt to start things off. Higher taxes keep future debt levels low but push the burden on to current generations; likewise, lower taxes today raise the debt level, pushing the burden on to future generations, and raising the likelihood of placing debt on an exploding path. Future generations benefit from a better environment and those welfare gains can be taxed without hurting them. The question is, is this tax revenue sufficient to prevent the debt level from exploding? Does the answer to this question constrain how ambitious the abatement policy can be? An additional challenge lies ahead. Can welfare improvements be delivered by the policy without necessitating cuts in consumption along the way?

Below, we study an environment-tax policy $\left\{\mu_{t+j}=\mu,\tau_{t+j},B_{t+j};\, j\geq 0\right\}$ that is inaugurated in period $t\left(j=0\right)$ of the BAU. (In places below, we consider the starting date of the policy to be the BAU steady state.) At the point the abatement policy is initiated, production is already complete and all emissions released. This implies the inaugural generation is unaffected by the policy -- their labor supply is predetermined from the BAU -- and hence they are not taxed. Consequently, the initial primary balance is $M_t=-\Lambda\left(\mu\right)G\left(L_t^{BAU}\right)<0$ and the initial debt needed to cover the abatement expense is positive. Thereafter, for $j>0$, the primary deficit evolves as $M_{t+j}=\tau_{t+j}L_{t+j}^{\mu}-\Lambda\left(\mu\right)G\left(L_{t+j}^{\mu}\right)$ where $L_{t+j}^{\mu}=L\left(S_{t+j}^{\mu},\tau_{t+j}\right)$ and the stock of bonds as $B_{t+j}=RB_{t+j-1}-M_{t+j}$. The law of motion for pollution is given by

$$S_{t+j+1}^{\mu}=\left(1-\varepsilon\right)S_{t+j}^{\mu}+\left(1-\mu\right)G\left(L\left(S_{t+j}^{\mu},\tau_{t+j}\right)\right).$$

Our next order of business is to compute the path of taxes, τ_{t+j}. Apropos our discussion above, we posit τ_{t+j} is chosen to satisfy the intergenerational Pareto criterion for $j > 0$:

$$\tilde{U}_{t+j}\left(c_{t+j+1}\left(S^{\mu}_{t+j},\tau_{t+j}\right),L\left(S^{\mu}_{t+j},\tau_{t+j}\right)\right) \geq \tilde{U}^{BAU}_{t}\left(c_{t+j+1}\left(S^{BAU}_{t+j}\right),L\left(S^{BAU}_{t+j}\right)\right), \tag{2.16}$$

where the l.h.s of (2.16), indirect utility under the government's environment-tax policy, is given by

$$\tilde{U}^{\mu}_{t+j}\left(c_{t+j+1},L_{t+j}\right) = U\left(\underbrace{R\left[H\left(S^{\mu}_{t+j}\right)F\left(L_{t+j}\left(S^{\mu}_{t+j},\tau_{t+j}\right)\right)-\tau_{t+j}L_{t+j}\left(S^{\mu}_{t+j},\tau_{t+j}\right)\right]}_{c_{t+j+1}},L_{t+j}\left(S^{\mu}_{t+j},\tau_{t+j}\right)\right) \tag{2.17}$$

and the r.h.s of (2.16) is the counterfactual, what indirect utility would have been had the BAU world persisted. Also note, since the inaugural generation is not taxed and all abatement at $j = 0$ is financed via debt, it is clear the inaugural generation enjoys the same utility as they would had the BAU continued.

Here we define the implementability set for μ. An abatement policy, μ, is implementable under the Pareto criterion if the associated tax rates τ_{t+j} satisfy (2.16), and the associated path of debt, satisfying (2.13) - (2.15) is non-exploding.

Moving forward, our goal is to characterize the path of taxes τ_{t+j} satisfying (2.16) and to specify the implementability set for μ. We also wish to provide an exact analysis of the dynamics of pollution under such a path of taxes and ascertain whether consumption is rising or falling along the transition.

2.4 Pareto-improving Environmental Policies under Quasi-linear Utility

With these goals in mind, we present a very special but tractable case of quasi-linear utility: $\sigma = 0$ (quasi-linear in consumption). More general specifications will be taken up in numerical examples below. Our starting point is the steady state under BAU meaning we assume the economy is already in the BAU steady state when the policy is first implemented. Our methodology allows for more generality, that is we can accommodate a setting in which the policy is inaugurated at any arbitrary point along the BAU transition. This last point is explored in numerical examples below.

In practice, it is impossible to simply solve the non-linear equation (2.16) for τ_{t+j} and compute a path of Pareto-improving taxes that way. A convenient strategy is to compute a path of taxes, call it $\widehat{\tau}_{t+j}$, such that one of the arguments of $\tilde{U}^{\mu}_{t+j}$, labor supply, is the same in the policy regime as in the BAU. Specifically, $\widehat{\tau}_{t+j}$ is that path of taxes that equates labor supply in the policy regime with that in the BAU -- i.e., $L^{\mu}_{t+j} = L^{*}_{BAU}$ where L^{*}_{BAU} is defined in (2.9). In Appendix 2.E, we show that for quasi-linear utility, $\sigma = 0$,

$$\widehat{\tau}_{t+j} = \left[H\left(S^{\mu}_{t+j}\right) - H\left(S^{*}_{BAU}\right)\right] F_L\left(L^{*}_{BAU}\right). \tag{2.18}$$

Using $\frac{F(.)}{L} > F_L(.)$, the same appendix also shows that $c^{\mu}_{t+j+1} > c^{*}_{BAU}$ implying for the path of taxes, all agents (from generation $t+1$ on) are strictly better off under the policy than they would have under the BAU.

Lemma 2.4 *In the special case of quasi-linear utility, an environmental-fiscal policy package,*

$$\left\{ \mu_{t+j} = \mu, \tau_{t+j} = \widehat{\tau}_{t+j} = \left[H\left(S^{\mu}_{t+j}\right) - H\left(S^{*}_{BAU}\right)\right] F_L\left(L^{*}_{BAU}\right), B_{t+j}; j \geq 0 \right\}$$

makes the utility of all generations (from generation $t+1$ *on) strictly higher than what they would have been had the BAU persisted. Under this package, the labor supply is the same pre and post policy while consumption is strictly higher.*

The following corollary argues that the result in the previous lemma can actually be extended to include all $\sigma < 1$, not just $\sigma = 0$.

Corollary 2.1 *More generally, if* $\sigma < 1$, *there exists a tax rate* $\widehat{\tau}_{t+j}$ *such that the environmental-fiscal policy package,*

$$\left\{ \mu_{t+j} = \mu, \tau_{t+j} = \widehat{\tau}_{t+j} < \left[H\left(S_{t+j}^{\mu}\right) - H\left(S_{BAU}^{*}\right) \right] \frac{F\left(L_{BAU}^{*}\right)}{L_{BAU}^{*}}, B_{t+j}; j \geq 0 \right\}$$

gives all generations (from generation $t+1$ *on) strictly higher utility than what they would have gotten had the BAU persisted. Under this package, the labor supply is the same pre and post policy while consumption is strictly higher.*

In passing, note that the condition on σ is a sufficient condition. That the implementability set for μ is non-empty for $\sigma > 1$ is explored in numerical examples below.

2.4.1 Dynamics of pollution and taxes

Since, by construction, the labor supplies are the same pre and post policy, $L_{t+j}^{\mu} = L_{BAU}^{*}$, the evolution equation for pollution can be written as

$$S_{t+j+1}^{\mu} = (1-\varepsilon) S_{t+j}^{\mu} + (1-\mu) G\left(L_{BAU}^{*}\right). \tag{2.19}$$

The dynamics of pollution (and everything else) becomes a lot more tractable since the emissions in the policy economy are given by $G\left(L_{BAU}^{*}\right)$ which is predetermined. After a bit of routine manipulation, it can be shown that

$$S^{\mu}_{t+j+1} = \left[\left(1-\varepsilon\right)^{j+1} \mu + \left(1-\mu\right) \right] S^{*}_{BAU} .$$

Clearly, S^{μ}_{t+j} is declining relative to S^{*}_{BAU} over time, and as $j \to \infty$, long-run pollution levels will approach $\left(1-\mu\right) S^{*}_{BAU}$, lower than its BAU counterpart. Declining pollution implies more output and more consumption; as such, under the Pareto criterion, taxes must rise. This is evident from $\widehat{\tau}_{t+j} = \left[H\left(S^{\mu}_{t+j}\right) - H\left(S^{*}_{BAU}\right) \right] F_L\left(L^{*}_{BAU}\right)$ and H is increasing since S^{μ}_{t+j} declines with j. These ideas are collected in the following proposition.

Proposition 2.2 *In the special case of quasi-linear utility, an environmental-fiscal policy package,*

$$\left\{ \mu_{t+j} = \mu, \tau_{t+j} = \widehat{\tau}_{t+j} = \left[H\left(S^{\mu}_{t+j}\right) - H\left(S^{BAU}_{t+j}\right) \right] F_L\left(L^{BAU}_{t+j}\right); j \geq 0 \right\}$$

is associated with a declining path of pollution relative to the BAU steady-state level. It is also associated with the same path of labor supply and a strictly increasing path of consumption relative to BAU levels. The path of taxes, $\tau_{t+j} = \widehat{\tau}_{t+j}$, *is increasing, however.*

Next, we turn to the all-important question, if the path of taxes is defined by $\widehat{\tau}_{t+j}$, what is the associated path for the public debt? Most importantly, does satisfying the Pareto criterion render the path explosive? In other words, is the implementability set for μ non-empty?

2.4.2 Path of debt

As discussed in Section 2.3.1, using $L^{\mu}_{t+j} = L^{*}_{BAU}$, the primary budget is given by

$$\begin{aligned} M_{t+j} &= \left(\widehat{\tau}_{t+j}\right) L^{\mu}_{t+j} - \Lambda\left(\mu\right) G\left(L^{\mu}_{t+j}\right) \\ &= \left[H\left(S^{\mu}_{t+j}\right) - H\left(S^{*}_{BAU}\right) \right] F_L\left(L^{*}_{BAU}\right) L^{*}_{BAU} - \Lambda\left(\mu\right) G\left(L^{*}_{BAU}\right) \end{aligned} \qquad (2.20)$$

implying the primary budget evolves as

$$M_{t+j+1} - M_{t+j} = \left[H\left(S^{\mu}_{t+j+1}\right) - H\left(S^{\mu}_{t+j}\right)\right] F_L\left(L^*_{BAU}\right) L^*_{BAU} > 0 \qquad (2.21)$$

Since S^{μ} has been established to be declining over time. Knowing $M_{t+j+1} - M_{t+j} > 0$ for all j, it follows from the evolution equation for debt $B_{t+j+1} - B_{t+j} = R\left(B_{t+j} - B_{t+j-1}\right) - \left(M_{t+j+1} - M_{t+j}\right)$ that if at some date k, $B_{t+k} - B_{t+k-1} < 0$, then $B_{t+j+1} - B_{t+j} < 0$ for all $j > k$. Debt levels, once they fall, continue to fall forever after. And because $R > 1$, the debt reaches 0 in finite time.

As discussed in Section 2.3.1, there are three critical dates of interest. First: when does M_{t+j} turn positive? The aforediscussed process of debt decline cannot start until a date when M_{t+j} turns positive, that is the government starts to run up a surplus of tax revenue over its abatement expenditures (after all, only then can it turn its attention to debt servicing). Second, when does $B_{t+j+1} - B_{t+j} < 0$ first happen? And third, when does $B_{t+j} \to 0$? We take these up in turn. For what we present below, we use the linear approximation for the damage function as in (2.2) for tractability's sake.

2.4.2.1 The first date when primary budget balance turns positive

Recall that the initial primary deficit, $M_t = -\Lambda(\mu) G\left(L^{BAU}_t\right) < 0$ and hence the initial debt needed to cover the abatement expense is positive. Thereafter the primary deficit evolves in a manner described by (2.20). Also recall that a necessary condition for the path of debt to decline is that M_{t+j} turns positive. Using (2.20) and (2.2), it is possible to write

$$M_{t+j} = \varepsilon \mu S^*_{BAU} \left(\frac{\rho F_L\left(L^*_{BAU}\right) L^*_{BAU}}{\varepsilon} \left(1 - (1-\varepsilon)^j\right) - \frac{\Lambda(\mu)}{\mu} \right)$$

Suppose the first date when M_{t+j} turns positive is $j = k_1$.

Lemma 2.5 *Suppose* $\mu \in \left[0, \overline{\mu}_1\right]$ *where* $1 - \frac{\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}} \frac{\Lambda\left(\overline{\mu}_1\right)}{\overline{\mu}_1} \equiv 0$. *Then,*

$$k_1 = \frac{\ln\left(1 - \frac{\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}} \frac{\Lambda(\mu)}{\mu}\right)}{\ln\left(1-\varepsilon\right)},$$

and hence the first date when M_{t+j} *turns positive is* $t + \lceil k_1 \rceil$ *where* $\lceil k_1 \rceil$ *is the smallest integer not less than* k_1. *Also, that first date rises as* μ *increases.*

2.4.2.2 The first date when the debt starts to decline: the debt turning point

The change in the level of debt between $t+j$ and $t+j+1$ is:

$$\begin{aligned}\Delta B_{t+j+1} &\equiv B_{t+j+1} - B_{t+j} \\ &= \left(RB_{t+j} - M_{t+j+1}\right) - \left(RB_{t+j-1} - M_{t+j}\right) = R\left(B_{t+j} - B_{t+j-1}\right) - \left(M_{t+j+1} - M_{t+j}\right)\end{aligned}$$

which upon repeated iteration yields

$$\Delta B_{t+j+1} = -\sum_{i=0}^{j} R^{j-i}\left(M_{t+i+1} - M_{t+i}\right) + R^{j+1}\left(B_t - B_{t-1}\right) = -R^{j+1}\left(\sum_{i=0}^{j} \frac{M_{t+i+1} - M_{t+i}}{R^{i+1}} + M_t\right)$$

Using (2.2), it is possible to write $M_{t+i+1} - M_{t+i} = \varepsilon\mu\rho S^*_{BAU} F_L\left(L^*_{BAU}\right)L^*_{BAU}\left(1-\varepsilon\right)^i$, which leads to

$$\Delta B_{t+j+1} = -R^{j+1}\varepsilon\mu S^*_{BAU}\left[\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{R-1+\varepsilon}\left(1 - \left(\frac{1-\varepsilon}{R}\right)^{j+1}\right) - \frac{\Lambda(\mu)}{\mu}\right]$$

Recall, on the first date $B_t > 0$ and also that once debt begins to fall, it falls forever. Noting that, there are three possibilities to consider: 1) the debt keeps increasing, 2) the debt first rises and then falls, and 3) the debt starts to decrease right away ($B_{t+1} < B_t$). In case (2) and (3), the debt will eventually reach zero. Suppose the first date when the debt declines is $j = k_2$.

Proposition 2.3 *Suppose* $\mu \in \left[0, \overline{\mu}_2 < \overline{\mu}_1\right]$ *where* $1 - \frac{R-1+\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}} \frac{\Lambda\left(\overline{\mu}_2\right)}{\overline{\mu}_2} \equiv 0$. *Then, as per case (2),*

$$k_2 = \frac{\ln\left(1 - \frac{R-1+\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}} \frac{\Lambda(\mu)}{\mu}\right)}{\ln\left(\frac{1-\varepsilon}{R}\right)}$$

and so the first date when the debt starts to decline is $t + \lceil k_2 \rceil$ *where* $\lceil k_2 \rceil$ *is the smallest integer not less than* k_2. *Also, that first date rises as* μ *increases.*

Note, if $\mu > \overline{\mu}_2$, case (1) obtains. For case (3) to obtain, we need $k_2 < 1$, which requires $\frac{\Lambda(\mu)}{\mu} < \frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{R}$ to hold.

2.4.2.3 The first date when debt levels reach zero

The debt in period $t+j$ (iterating $B_{t+j} = RB_{t+j-1} - M_{t+j}$ and using $B_{t-1} = 0$) is given by

$$B_{t+j} = -R^j \sum_{i=0}^{j} \frac{M_{t+i}}{R^i}.$$

Using (2.2), it is possible to write

$$B_{t+j} = -\varepsilon\mu S^*_{BAU} \frac{R^{j+1}-1}{R-1}\left(\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{\varepsilon}\left(1 - \frac{R-1}{R-1+\varepsilon}\frac{1-\left(\frac{1-\varepsilon}{R}\right)^{j+1}}{1-\left(\frac{1}{R}\right)^{j+1}}\right) - \frac{\Lambda\left(\mu\right)}{\mu}\right)$$

Recall $B_t = \varepsilon S^*_{BAU}\Lambda\left(\mu\right) > 0$ and $B_{t+1} = -\varepsilon\mu S^*_{BAU}\left(1+R\right)\left(\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{1+R} - \frac{\Lambda(\mu)}{\mu}\right)$. If $\frac{\Lambda(\mu)}{\mu} < \frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{1+R}$, then $B_{t+1} < 0$, meaning the debt turns negative from the second period itself. It is a subset of the range of μ which makes debt decline from the second period ($\frac{\Lambda(\mu)}{\mu} < \frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{R}$). Based on the analysis above, we know in case (2) and (3), the debt will reach zero in finite time. Suppose the date at which the debt hits zero is $j = k_3$.

Proposition 2.4 *k_3 is the solution to*

$$\frac{\rho F_L\left(L_{BAU}^*\right)L_{BAU}^*}{\varepsilon}\left(1-\frac{R-1}{R-1+\varepsilon}\frac{1-\left(\frac{1-\varepsilon}{R}\right)^{k_3+1}}{1-\left(\frac{1}{R}\right)^{k_3+1}}\right)=\frac{\Lambda(\mu)}{\mu} \tag{2.22}$$

To ensure (2.22) has a solution $k_3 > 0$, we require

$$\frac{\Lambda(\mu)}{\mu} < \frac{\rho F_L\left(L_{BAU}^*\right)L_{BAU}^*}{R-1+\varepsilon},$$

the same range of μ that is required for the existence of the debt turning point in Proposition 2.3. The debt is zero at date $t + \lceil k_3 \rceil$ and that date is pushed back if a higher μ is to be implemented.

The upshot is that the implementability set for μ is non-empty, that there exists a range of μ that the government can usher in under a generational Pareto criterion that leaves every generation at least as well off, possibly better, than if the BAU world had continued. The policy economy has lower pollution and higher consumption as well.

2.5 Robustness Checks

In this section, we study the robustness of some of our results to some alternative formulations. First and foremost, we wish to demonstrate that the general tenor of our results go through when $\sigma > 1$. Second, we check if the policy can be inaugurated at any point in the BAU transition, not necessarily at the steady state. Third, we had, for tractability's sake assumed an affine damage function in the computation of the debt dynamics; here, we relax that restriction as well. And finally, we allow emissions to vary with output as in Karp and Rezai (2014b).

Our model is designed to offer qualitative insight, and pursuant to that end, is silent on the quantitative margin. The goal here is not a full-blown calibration exercise but rather to paint a picture of the Pareto-improving transition with broad brushstrokes to see if environmental policy can improve matters and the associated debt paths don't misbehave. We start by assigning parameter values that are in line with established practice in the literature. The following functional forms are used:

Utility: $U_{t+j}\left(c_{t+j+1}, L_{t+j}\right) = \frac{c_{t+j+1}^{1-\sigma}}{1-\sigma} - \beta \frac{L_{t+j}^{1-\gamma}}{1-\gamma};\ \sigma \geq 0,\ \beta > 0,\ \gamma < 0.$

Production: $F\left(L_{t+j}\right) = AL_{t+j}^{\alpha};\ A > 0,\ \alpha < 1.$

Damage: $H\left(S_{t+j}\right) = \frac{1}{1+\rho S_{t+j}^{2}};\ \rho > 0.$

Abatement cost: $\Lambda\left(\mu\right) = \lambda \mu^{\varphi};\ \lambda > 0,\ \varphi > 1.$

Emission: $e_{t+j} = \left(1-\mu\right)\delta y_{t+j};\ \delta > 0.$

The parameters of the model are chosen as follows: $\sigma = 1.7$, $\gamma = -0.5$, $\beta = 0.1$, $A = 100$, $\alpha = 0.6$, $\rho = 4\times10^{-7}$, $\lambda = 0.054$, $\phi = 2.8$, $\delta = 0.062$, $R = 2$ and $\varepsilon = 0.126$. The functional forms for damage $H(\cdot)$, abatement cost $\Lambda(\cdot)$ and emissions e and corresponding parameter values (ρ, λ, φ, δ and ε) are in line with those used in Karp and Rezai (2014b). By considering the length of each period as 35 years, the exogenous interest rate is calculated as $R = (1+0.02)^{35} = 2$. A is a scale parameter and α is chosen to make labor's share of output (since we do not have capital in the model) equal to 0.6. A small β is chosen so as to not make our results too reliant on a strong labor supply response.

We consider the following environmental policy: the government starts to abate 20% of the emissions generated in each period (hence, $\mu = 0.2$) starting from $t = 0$. We consider two cases: (i) the policy is inaugurated from the BAU steady state (Figure 2.2); (ii) the policy is inaugurated from a date well before the steady state is reached; specifically, it starts at a point where the pollution stock is lower than that at the BAU steady state. More specifically, we set $S_{t=0} = 0.85 \times S_{BAU}$ (Figure 2.3). In each case, we choose a sequence $\left\{\varepsilon_{t+j} > 0\right\}_{j\geq 1}$ and make $U_{t+j}(\cdot) = U_{t+j}^{BAU}(\cdot) + \varepsilon_{t+j}$ for all $j > 0$, that is, all generations from $t+1$ are strictly better off.[21] In both cases, the debt is paid off in about 12 periods and compared with the BAU, agents along the transition not only have higher utility, they also work less and consume more -- unlike much of the literature, we don't require them to sacrifice either consumption or leisure to make them happier.

2.6 Conclusions

This paper studies a tractable small open economy populated by overlapping generations of agents facing a standard stock externality from pollution caused by productive activities. In the laissez faire equilibrium, environmental quality gets worse over time, and consumption and utility falls. The business-as-usual situation is a grim one and presents an opportunity for government intervention in the form of pollution abatement. The catch is that such policies are costly and it

[21] As explained earlier, the way we've set things up, the generation born at the inaugural date cannot benefit from this environmental policy, unless the government borrows to makes transfers to them (which we disallow). The inaugural generations' utility is held at the BAU level; all others are made strictly better off.

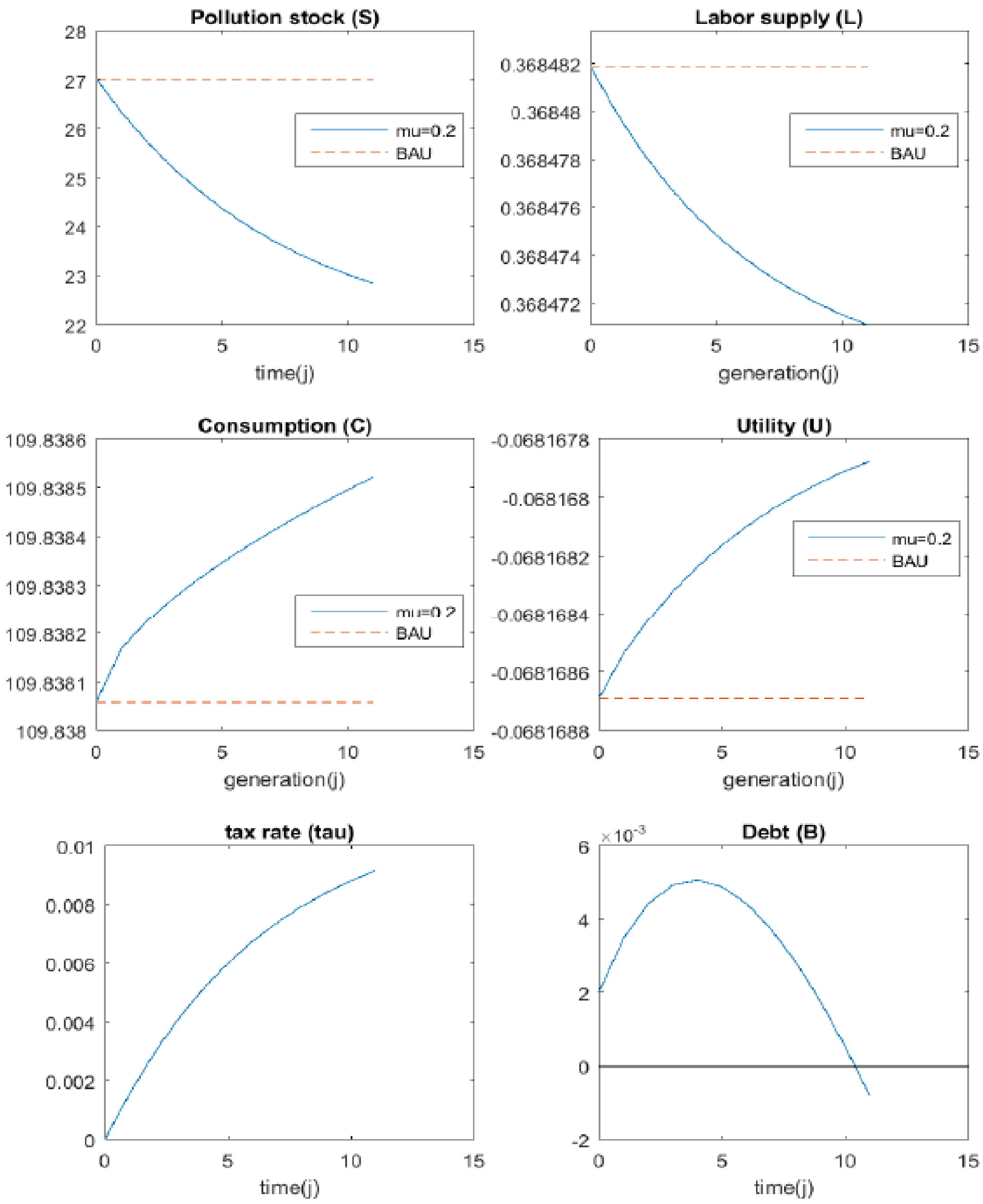

Figure 2.2. *Starting from the BAU steady state*

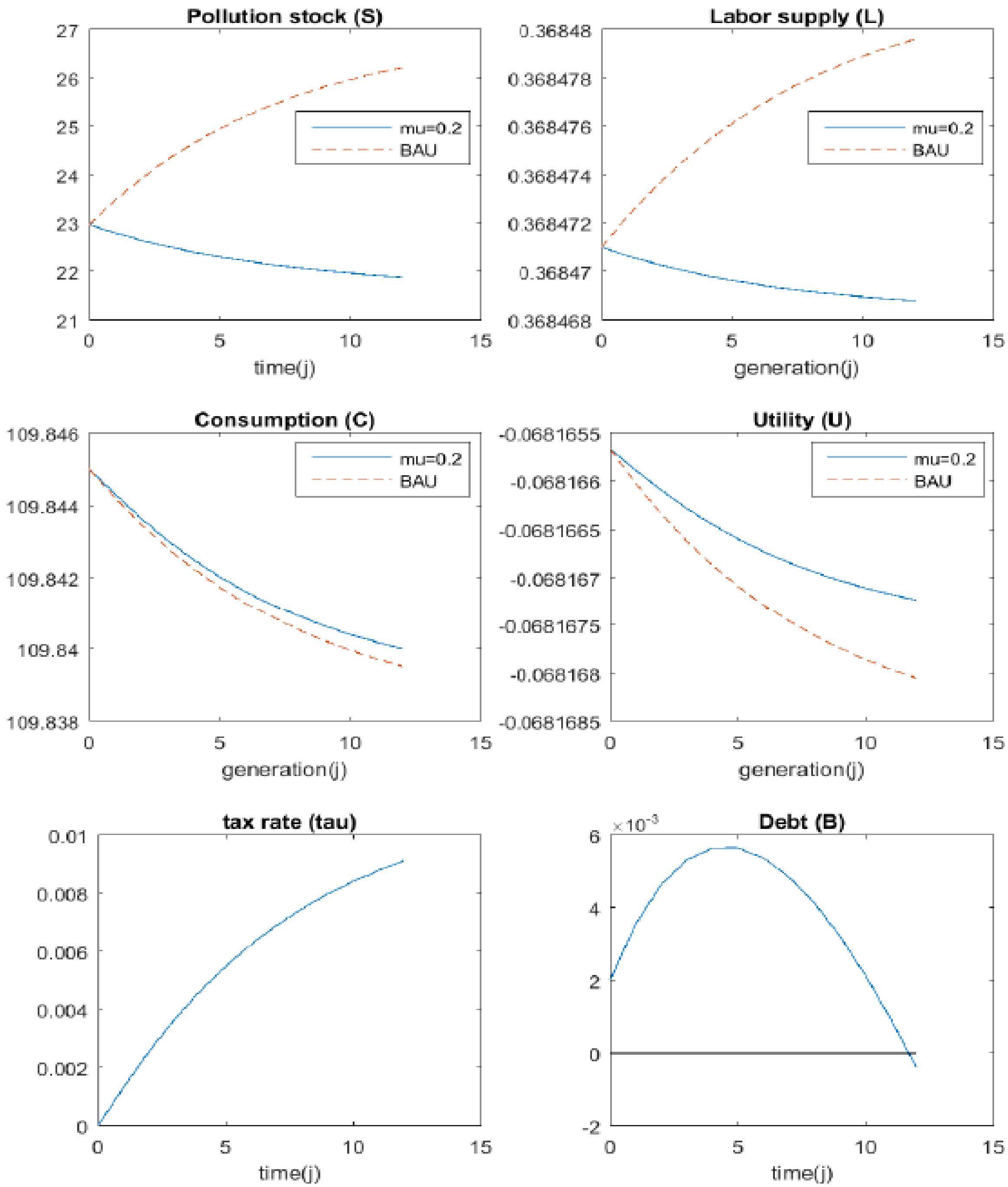

Figure 2.3. *Starting from a point along the BAU transition*

takes a while for the benefits to start appearing in a substantial way. The government can borrow on international markets to start the abatement and can tax some of the downstream welfare gains to help pay down the debt. The big question is, can the government usher in such an environmental policy that makes sure that no generation is hurt (indeed all are better off) and the debt is paid off in finite time? We show, the answer is in the affirmative. The new equilibrium has lower pollution levels than in the business as usual world. Along the transition, every generation is better off (at least no worse off) in utility terms and consumption is also rising.

Two additional points are worth noting. First, how does our discussion change if the assumption of a small open economy is abandoned in favor of a closed economy (with neoclassical production). With neoclassical production, factor prices are endogenous and the effects of policy choices at the initial date will, via its effects on endogenous variables such as saving (and hence capital stock, and factor returns), will linger forever. It is apparent that debt will crowd out private saving thereby reducing the capital stock. This causes the wage rate to decrease and the interest rate to increase. Whether these effects ease the implementation hurdles at that date, at future dates, is not at all clear. The dynamics of debt becomes immensely complicated since it gets coupled with the dynamics of the endogenously-evolving capital stock. It is our conjecture that the sorts of effects we discuss in the current paper will continue to operate in this more complicated setting.

Our analysis has also stayed away from studying alternative policies that put a direct cap on labor supply (through mandatory length of work week laws). Also, instead of using debt, the generations could work out a corresponding path of intergenerational transfers as in von Below et al. (2015). It is our conjecture that any attempt to introduce such policies under the Pareto criterion would presumably face similar implementation hurdles as raised here.

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APPENDIX 2.A

PROOF OF LEMMA 2.1

From the first order condition (2.4), we can calculate

$$£_{LL} = u_{cc}(\cdot) R^2 \left[H(S_{t+j}) F_L(L_{t+j}) - \tau_{t+j} \right]^2 + u_c(\cdot) R H(S_{t+j}) F_{LL}(L_{t+j}) + v_{ll}(1 - L_{t+j}) < 0,$$

$$\begin{aligned} £_{LS} &= u_{cc}(\cdot) R H_S(S_{t+j}) F(L_{t+j}) R \left[H(S_{t+j}) F_L(L_{t+j}) - \tau_{t+j} \right] + u_c(\cdot) R H_S(S_{t+j}) F_L(L_{t+j}) \\ &= u_c(\cdot) R H_S(S_{t+j}) F_L(L_{t+j}) \left[1 - \frac{-c_{t+j+1} u_{cc}(\cdot)}{u_c(\cdot)} \frac{H(S_{t+j}) F(L_{t+j}) F_L(L_{t+j}) - \tau_{t+j} F(L_{t+j})}{H(S_{t+j}) F(L_{t+j}) F_L(L_{t+j}) - \tau_{t+j} F_L(L_{t+j}) L_{t+j}} \right], \end{aligned}$$

and

$$\begin{aligned} £_{L\tau} &= -u_{cc}(\cdot) R L_{t+j} R \left[H(S_{t+j}) F_L(L_{t+j}) - \tau_{t+j} \right] - u_c(\cdot) R \\ &= -u_c(\cdot) R \left[1 - \frac{-c_{t+j+1} u_{cc}(\cdot)}{u_c(\cdot)} \frac{H(S_{t+j}) F_L(L_{t+j}) L_{t+j} - \tau_{t+j} L_{t+j}}{H(S_{t+j}) F(L_{t+j}) - \tau_{t+j} L_{t+j}} \right]. \end{aligned}$$

It follows that

$$\frac{\partial L_{t+j}}{\partial S_{t+j}} = -\frac{£_{LS}}{£_{LL}}, \frac{\partial L_{t+j}}{\partial \tau_{t+j}} = -\frac{£_{L\tau}}{£_{LL}}.$$

In the BAU, $\tau = 0$, and then $\frac{\partial L_{t+j}}{\partial S_{t+j}}$ can be simplified to

$$\left. \frac{\partial L_t}{\partial S_t} \right|_{BAU} = -\frac{u_c(\cdot) R H_S(S_t) F_L(L_t) \left[1 - \left(\frac{-c_{t+1} u_{cc}(\cdot)}{u_c(\cdot)} \right) \right]}{£_{LL}}.$$

The rest follows.

APPENDIX 2.B

PROOF OF LEMMA 2.2

$$\begin{aligned}
\frac{\partial c_{t+j+1}}{\partial S_{t+j}} &= R\left[H_S\left(S_{t+j}\right)F\left(L_{t+j}\right)+H\left(S_{t+j}\right)F_L\left(L_{t+j}\right)\frac{\partial L_{t+j}}{\partial S_{t+j}}-\tau_{t+j}\frac{\partial L_{t+j}}{\partial S_{t+j}}\right] \\
&= R\left[H_S\left(S_{t+j}\right)F\left(L_{t+j}\right)+\left(H\left(S_{t+j}\right)F_L\left(L_{t+j}\right)-\tau_{t+j}\right)\frac{\partial L_{t+j}}{\partial S_{t+j}}\right] \\
&= R\left[H_S\left(S_{t+j}\right)F\left(L_{t+j}\right)-\left(H\left(S_{t+j}\right)F_L\left(L_{t+j}\right)-\tau_{t+j}\right)\frac{u_{cc}(\cdot)RH_S(\cdot)F(\cdot)R\left[H(\cdot)F_L(\cdot)-\tau_{t+j}\right]+u_c(\cdot)RH_S(\cdot)F_L(\cdot)}{u_{cc}(\cdot)R^2\left[H(\cdot)F_L(\cdot)-\tau_{t+j}\right]^2+u_c(\cdot)RH(\cdot)F_{LL}(\cdot)+v_{ll}(\cdot)}\right] \\
&= RH_S\left(S_{t+j}\right)F\left(L_{t+j}\right)\frac{u_c(\cdot)RH\left(S_{t+j}\right)F_{LL}\left(L_{t+j}\right)-\frac{F_L\left(L_{t+j}\right)}{F\left(L_{t+j}\right)}\underbrace{u_c(\cdot)R\left(H(\cdot)F_L(\cdot)-\tau_{t+j}\right)}_{=v_l(\cdot)>0}+v_{ll}(\cdot)}{u_{cc}(\cdot)R^2\left[H(\cdot)F_L(\cdot)-\tau_{t+j}\right]^2+u_c(\cdot)RH(\cdot)F_{LL}(\cdot)+v_{ll}(\cdot)} \\
&< 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial c_{t+j+1}}{\partial \tau_{t+j}} &= R\left[H\left(S_{t+j}\right)F_L\left(L_{t+j}\right)\frac{\partial L_{t+j}}{\partial \tau_{t+j}}-L\left(S_{t+j},\tau_{t+j}\right)-\tau_{t+j}\frac{\partial L_{t+j}}{\partial \tau_{t+j}}\right] \\
&= R\left[\left(H\left(S_{t+j}\right)F_L\left(L_{t+j}\right)-\tau_{t+j}\right)\frac{\partial L_{t+j}}{\partial \tau_{t+j}}-L\left(S_{t+j},\tau_{t+j}\right)\right] \\
&= R\left[-\left(H\left(S_{t+j}\right)F_L\left(L_{t+j}\right)-\tau_{t+j}\right)\times\frac{-u_{cc}(\cdot)RL_{t+j}R\left[H(\cdot)F_L(\cdot)-\tau_{t+j}\right]-u_c(\cdot)R}{u_{cc}(\cdot)R^2\left[H(\cdot)F_L(\cdot)-\tau_{t+j}\right]^2+u_c(\cdot)RH(\cdot)F_{LL}(\cdot)+v_{ll}(\cdot)}-L\left(S_{t+j},\tau_{t+j}\right)\right] \\
&= RL_{t+j}\left[\frac{u_c(\cdot)R\left(H(\cdot)F_L(\cdot)-\tau_{t+j}\right)-u_c(\cdot)RH(\cdot)F_{LL}(\cdot)-v_{ll}(\cdot)}{u_{cc}(\cdot)R^2\left[H(\cdot)F_L(\cdot)-\tau_{t+j}\right]^2+u_c(\cdot)RH(\cdot)F_{LL}(\cdot)+v_{ll}(\cdot)}\right] \\
&< 0
\end{aligned}$$

APPENDIX 2.C

PROOF OF LEMMA 2.3

By the envelope theorem, we have

$$\frac{\partial \tilde{U}_{t+j}}{\partial S_{t+j}} = RU_c(\cdot)H_s(\cdot) < 0, \text{ and } \frac{\partial \tilde{U}_{t+j}}{\partial \tau_{t+j}} = -RU_c(\cdot)L_{t+j} < 0.$$

APPENDIX 2.D

PROOF OF PROPOSITION 2.1

$$\frac{\partial U^{SP}}{\partial L} = u_c\left(RH\left(\frac{G(L^{SP})}{\varepsilon}\right)F(L^{SP})\right)R\left[\underbrace{H_S(.)\frac{G_L(L^{SP})}{\varepsilon}F(L^{SP})}+H\left(\frac{G(L^{SP})}{\varepsilon}\right)F_L(L^{SP})\right]-v_l(1-L^{SP})$$

When evaluated at L^P, we have using the first order condition to the agent's problem,

$$\left.\frac{\partial U^{SP}}{\partial L}\right|_{L^P} = u_c(\cdot)R\left[H_S(.)\frac{G_L(L^{SP})}{\varepsilon}F(L^{SP})\right] < 0$$

implying $L^P > L^{SP}$.

APPENDIX 2.E

PATH OF TAXES UNDER POLICY-INVARIANT LABOR SUPPLY

As discussed above, the inaugural generation $t(j=0)$ is unaffected by the policy. Because of the government's abatement activity during period t, the start-of-period stock of pollution next period ($j=1$) satisfies $S_{t+1}^{\mu} < S_{t+1}^{BAU}$. If the government imposes no taxes, then it follows from Lemma 2.3 that generation $t+1$ will be strictly better off. Some or all of this welfare gain may be taxed away by the government to help defray (part of) the abatement and debt service costs in that period. There exists a range for the tax rate, say $\tau_{t+1} \in \left[0, \overline{\tau}_{t+1}\right]$ such that $\tilde{U}_{t+1}\left(S_{t+1}^{\mu}, \tau_{t+1}\right) \geq \tilde{U}_{t+1}^{BAU}\left(S_{t+1}^{BAU}\right)$. When $\tau_{t+1} = \overline{\tau}_{t+1}$, $\tilde{U}_{t+1} = \tilde{U}_{t+1}^{BAU}$ (a Pareto-neutral choice of tax) and if $\tau_{t+1} = 0$, the government leaves all the welfare gains to generation $t+1$.

For now, we set aside our search for Pareto-improving taxes. Instead, we focus on a subset of taxes $\left\{\hat{\tau}_{t+j}\right\}_{j=1}^{\infty}$ such that $L_{t+j}^{\mu} = L\left(S_{t+j}^{\mu}, \hat{\tau}_{t+j}\right) = L\left(S_{t+j}^{BAU}\right), \forall j \geq 0$. In other words, $\hat{\tau}_{t+j}$ is chosen to keep labor supply under the government's policy the same as its level in the BAU. This helps fix the second argument of $\tilde{U}\left(.,.\right)$ in (2.16). We wish to investigate what implication this may have for the first argument, consumption, and via this channel, dig deeper into (2.16). The associated debt dynamics are a separate matter which we will turn to further below.

Start with the optimality conditions for labor supply, pre and post policy, and use them to back out the necessary path of taxes using the fact $L_{t+j}^{\mu} = L\left(S_{t+j}^{BAU}\right)$. This means

$$\text{BAU: } u_c\left(\underbrace{RH\left(S_{t+j}^{BAU}\right)F\left(L_{t+j}^{BAU}\right)}_{c_{t+j+1}^{BAU}}\right)RH\left(S_{t+j}^{BAU}\right)F_L\left(L_{t+j}^{BAU}\right) - v_l\left(1 - L_{t+j}^{BAU}\right) = 0$$

$$\text{Policy: } U_c\left(\underbrace{R\left[H\left(S_{t+j}^{\mu}\right)F\left(L_{t+j}^{\mu}\right) - \hat{\tau}_{t+j}L_{t+j}^{\mu}\right]}_{c_{t+j+1}^{\mu}}\right)R\left[H\left(S_{t+j}^{\mu}\right)F_L\left(L_{t+j}^{\mu}\right) - \hat{\tau}_{t+j}\right] - v_l\left(1 - L_{t+j}^{\mu}\right) = 0$$

with $L_{t+j}^{\mu} = L_{t+j}^{BAU}$.

For (2.3), these equations reduce to

$$\left(\frac{H\left(S_{t+j}^{\mu}\right)}{H\left(S_{t+j}^{BAU}\right)}-\frac{\hat{\tau}_{t+j}}{H\left(S_{t+j}^{BAU}\right)\frac{F\left(L_{t+j}^{BAU}\right)}{L_{t+j}^{BAU}}}\right)^{\sigma}=\frac{H\left(S_{t+j}^{\mu}\right)}{H\left(S_{t+j}^{BAU}\right)}-\frac{\hat{\tau}_{t+j}}{H\left(S_{t+j}^{BAU}\right)F_L\left(L_{t+j}^{BAU}\right)}. \quad (2.23)$$

In general, $\frac{F(L_t)}{L_t} > F_L\left(L_t\right)$ holds. First note, when $\sigma = 1$, there does not exist $\hat{\tau}_{t+j} > 0$ satisfying (2.23) and therefore, labor supply pre and post policy cannot be the same. This is because with a logarithmic utility, a better environment has no direct effect on the labor supply. If the government nevertheless collects taxes anyway, labor supply would change, and utility cannot be brought back to BAU levels.

If $\sigma \neq 1$, then with $\hat{\tau}_{t+j} > 0$, eq. (2.23) implies

$$\left(\frac{H\left(S_{t+j}^{\mu}\right)}{H\left(S_{t+j}^{BAU}\right)}-\frac{\hat{\tau}_{t+j}}{H\left(S_{t+j}^{BAU}\right)\frac{F_L\left(L_{t+j}^{BAU}\right)}{L_{t+j}^{BAU}}}\right)^{\sigma-1}<1. \quad (2.24)$$

If $\sigma < 1$, then it follows from (2.24) that $\hat{\tau}_{t+j}$ must satisfy

$$\hat{\tau}_{t+j} < \frac{F\left(L_{t+j}^{BAU}\right)}{L_{t+j}^{BAU}}\left[H\left(S_{t+j}^{\mu}\right)-H\left(S_{t+j}^{BAU}\right)\right],$$

in which case

$$c_{t+j+1}^{\mu} = R\left[H\left(S_{t+j}^{\mu}\right)F\left(L_{t+j}^{BAU}\right)-\hat{\tau}_{t+j}L_{t+j}^{BAU}\right] > RH\left(S_{t+j}^{BAU}\right)F\left(L_{t+j}^{BAU}\right)=c_{t+j+1}^{BAU} \quad (2.25)$$

must hold. This means tax rates $\hat{\tau}_{t+j}$ that keep labor supply unchanged pre and post policy will benefit agents in consumption terms and offer higher utility relative to what they would get in the BAU. In general, we cannot solve $\hat{\tau}_{t+j}$ explicitly from (2.23). In a special case with quasi-linear utility function ($\sigma = 0$), we can solve $\hat{\tau}_{t+j} = \left[H\left(S_{t+j}^{\mu}\right)-H\left(S_{t+j}^{BAU}\right)\right]F_L\left(L_{t+j}^{BAU}\right)$.

APPENDIX 2.F

PROOF OF LEMMA 2.5

If we use a linear approximation for damage function as in (2.2), then

$$
\begin{aligned}
M_{t+j} &= \rho\left(S_B^* - S_{t+j}\right)F_L\left(L_B^*\right)L_B^* - \Lambda(\mu)G\left(L_B^*\right) \\
&= \mu\rho S_B^* F_L\left(L_B^*\right)L_B^*\left[1-(1-\varepsilon)^j\right] - \Lambda(\mu)\varepsilon S_B^* \\
&= \varepsilon\mu S_B^*\left(\frac{\rho F_L\left(L_B^*\right)L_B^*}{\varepsilon}\left(1-(1-\varepsilon)^j\right) - \frac{\Lambda(\mu)}{\mu}\right)
\end{aligned}
$$

Because $M_t < 0$ and M_{t+j} is increasing over time, to calculate the first date when it turns positive, we solve $M_{t+k_1} = 0$ and get

$$1 - \tfrac{\varepsilon}{\rho F_L\left(L_B^*\right)L_B^*}\tfrac{\Lambda(\mu)}{\mu} > 0.$$

First notice that only when $1 - \tfrac{\varepsilon}{\rho F_L\left(L_B^*\right)L_B^*}\tfrac{\Lambda(\mu)}{\mu} > 0$ can we have a solution for k_1. This sets an upper bound for μ (because $\tfrac{\Lambda(\mu)}{\mu}$ is increasing in μ). Within this range $\left[0, \overline{\mu}_1\right]$, we can solve

$$k_1 = \frac{\ln\left(1 - \tfrac{\varepsilon}{\rho F_L\left(L_B^*\right)L_B^*}\tfrac{\Lambda(\mu)}{\mu}\right)}{\ln(1-\varepsilon)}.$$

We get the first date when M turns positive ($k^{M>0}$) by rounding up (taking the ceiling of k_1): $k^{M>0} = t + \lceil k_1 \rceil$. It can be easily shown $\frac{dk^{M>0}}{d\mu} > 0$.

APPENDIX 2.G

PROOF OF PROPOSITION 2.3

Use (2.2) in

$$\Delta B_{t+j+1} = -\sum_{i=0}^{j} R^{j-i}\left(M_{t+i+1}-M_{t+i}\right)+R^{j+1}\left(B_t - B_{t-1}\right) = -R^{j+1}\left(\sum_{i=0}^{j}\frac{M_{t+i+1}-M_{t+i}}{R^{i+1}}+M_t\right)$$

noting $M_{t+i+1}-M_{t+i} = \varepsilon\mu\rho S^*_{BAU}F_L\left(L^*_{BAU}\right)L^*_{BAU}\left(1-\varepsilon\right)^i$, to get

$$\begin{aligned}\Delta B_{t+j+1} &= -R^{j+1}\left(\sum_{i=0}^{j}\frac{\varepsilon\mu\rho S^*_{BAU}F_L\left(L^*_{BAU}\right)L^*_{BAU}\left(1-\varepsilon\right)^i}{R^{i+1}} - \varepsilon\mu S^*_B\frac{\Lambda(\mu)}{\mu}\right)\\ &= -R^{j+1}\varepsilon\mu S^*_{BAU}\left[\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{R-1+\varepsilon}\left(1-\left(\frac{1-\varepsilon}{R}\right)^{j+1}\right)-\frac{\Lambda(\mu)}{\mu}\right]\end{aligned}$$

We have shown that $B_t > 0$ and once debt begins to fall, it will fall forever and reach zero in finite periods. To find the first date when debt declines, $k^{\Delta B<0}$, we solve $\Delta B_{t+k_2} = 0$:

$$\left(\frac{1-\varepsilon}{R}\right)^{k_2} = 1-\frac{R-1+\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}\frac{\Lambda(\mu)}{\mu}$$

Only when $1-\frac{R-1+\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}\frac{\Lambda(\mu)}{\mu} > 0$ can we have a solution for k_2. This sets an upper bound for μ, $\overline{\mu}_2$. Since $R > 1$, $\overline{\mu}_2 < \overline{\mu}_1$. This is consistent with our understanding that a necessary condition for debt decline is to have a positive M at an earlier date. Within the range $\left[0, \overline{\mu}_2\right]$, we can solve

$$k_2 = \frac{\ln\left(1-\frac{R-1+\varepsilon}{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}\frac{\Lambda(\mu)}{\mu}\right)}{\ln\left(\frac{1-\varepsilon}{R}\right)}.$$

Thus $k^{\Delta B<0} = t + \lceil k_2 \rceil$. It can be easily shown $\frac{dk^{\Delta B<0}}{d\mu} > 0$.

APPENDIX 2.H

PROOF OF PROPOSITION 2.4

The debt in period $t+j$ is (iterating $B_{t+j} = RB_{t+j-1} - M_{t+j}$ and using $B_{t-1} = 0$) is given by

$$B_{t+j} = -R^j \sum_{i=0}^{j} \frac{M_{t+i}}{R^i}.$$

With a linear damage function,

$$\begin{aligned} B_{t+j} &= -R^j \sum_{i=0}^{j} \frac{\varepsilon\mu S^*_{BAU}\left(\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{\varepsilon}\left(1-\left(1-\varepsilon\right)^i\right)-\frac{\Lambda(\mu)}{\mu}\right)}{R^i} \\ &= -\varepsilon\mu S^*_B \frac{R^{j+1}-1}{R-1}\left(\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{\varepsilon}\left(1-\frac{R-1}{R-1+\varepsilon}\frac{1-\left(\frac{1-\varepsilon}{R}\right)^{j+1}}{1-\left(\frac{1}{R}\right)^{j+1}}\right)-\frac{\Lambda(\mu)}{\mu}\right) \end{aligned}$$

To find the first date when debt reaches zero, $k^{B<0}$, we solve $B_{t+k_3} = 0$:

$$\frac{\rho F_L\left(L^*_{BAU}\right)L^*_{BAU}}{\varepsilon}\left(1-\frac{R-1}{R-1+\varepsilon}\frac{1-\left(\frac{1-\varepsilon}{R}\right)^{k_3+1}}{1-\left(\frac{1}{R}\right)^{k_3+1}}\right)=\frac{\Lambda(\mu)}{\mu}.$$

Define $z(x) \equiv \frac{1-\left(\frac{1-\varepsilon}{R}\right)^{x+1}}{1-\left(\frac{1}{R}\right)^{x+1}}, x>0$.

$$\begin{aligned} \frac{dz(x)}{dx} &= \frac{-\left(\frac{1-\varepsilon}{R}\right)^{x+1}\left[1-\left(\frac{1}{R}\right)^{x+1}\right]\ln\left(\frac{1-\varepsilon}{R}\right)+\left(\frac{1}{R}\right)^{x+1}\left[1-\left(\frac{1-\varepsilon}{R}\right)^{x+1}\right]\ln\left(\frac{1}{R}\right)}{\left[1-\left(\frac{1}{R}\right)^{x+1}\right]^2} \\ &= -\left(\frac{1-\varepsilon}{R}\right)^{x+1}\frac{\left[1-\left(\frac{1}{R}\right)^{x+1}\right]\ln\left(\frac{1-\varepsilon}{R}\right)-\left[\left(\frac{1}{1-\varepsilon}\right)^{x+1}-\left(\frac{1}{R}\right)^{x+1}\right]\ln\left(\frac{1}{R}\right)}{\left[1-\left(\frac{1}{R}\right)^{x+1}\right]^2} \\ &= -\left(\frac{1-\varepsilon}{R}\right)^{x+1}\frac{-\left[1-\left(\frac{1}{R}\right)^{x+1}\right]\ln\left(\frac{1}{1-\varepsilon}\right)+\left[1-\left(\frac{1}{1-\varepsilon}\right)^{x+1}\right]\ln\left(\frac{1}{R}\right)}{\left[1-\left(\frac{1}{R}\right)^{x+1}\right]^2} \end{aligned}$$

Define the numerator as $Q(x) \equiv -\left[1-\left(\frac{1}{R}\right)^{x+1}\right]\ln\left(\frac{1}{1-\varepsilon}\right)+\left[1-\left(\frac{1}{1-\varepsilon}\right)^{x+1}\right]\ln\left(\frac{1}{R}\right), x \geq 0$.

$$\frac{dQ(x)}{dx} = \ln\left(\frac{1}{1-\varepsilon}\right)\left(\frac{1}{R}\right)^{x+1}\ln\left(\frac{1}{R}\right) - \ln\left(\frac{1}{R}\right)\left(\frac{1}{1-\varepsilon}\right)^{x+1}\ln\left(\frac{1}{1-\varepsilon}\right)$$

$$= \underbrace{\ln\left(\frac{1}{1-\varepsilon}\right)}_{>0}\underbrace{\ln\left(\frac{1}{R}\right)}_{<0}\left[\underbrace{\left(\frac{1}{R}\right)^{x+1} - \left(\frac{1}{1-\varepsilon}\right)^{x+1}}_{<0}\right] > 0$$

so $Q(x)$ is increasing in x for $x \geq 0$.

$$Q(0) = -\left(1-\frac{1}{R}\right)\ln\left(\frac{1}{1-\varepsilon}\right)+\left(1-\frac{1}{1-\varepsilon}\right)\ln\left(\frac{1}{R}\right) = \left(\underbrace{1-\frac{1}{1-\varepsilon}}_{<0}\right)\left(\underbrace{1-\frac{1}{R}}_{>0}\right)\left(\underbrace{\frac{\ln\left(\frac{1}{R}\right)}{1-\frac{1}{R}} - \frac{\ln\left(\frac{1}{1-\varepsilon}\right)}{1-\frac{1}{1-\varepsilon}}}_{<0}\right) > 0$$

The last term is negative because $\frac{\ln x}{1-x}$ is increasing in x and $\frac{1}{R} < \frac{1}{1-\varepsilon}$.

$\frac{dQ(x)}{dx} > 0$ and $Q(0) > 0$, so $Q(x) > 0$, and hence $\frac{dz(x)}{dx} < 0$. The monotonicity implies

$$\frac{\rho F_L\left(L_{BAU}^*\right)L_{BAU}^*}{\varepsilon}\left(1-\frac{R-1}{R-1+\varepsilon}\frac{1-\left(\frac{1-\varepsilon}{R}\right)^{k_3+1}}{1-\left(\frac{1}{R}\right)^{k_3+1}}\right) \in \left(0, \frac{\rho F_L\left(L_{BAU}^*\right)L_{BAU}^*}{R-1+\varepsilon}\right)$$

To ensure k_3 exists, we require $\frac{\Lambda(\mu)}{\mu} < \frac{\rho F_L\left(L_{BAU}^*\right)L_{BAU}^*}{R-1+\varepsilon}$, the same range for the existence of a turning point. $k^{B<0} = t + \lceil k_3 \rceil$. Also, since $\frac{dz(x)}{dx} < 0$, it can be easily shown $\frac{dk^{B<0}}{d\mu} > 0$.

CHAPTER 3

OIL PRICES, EXCHANGE RATES AND THE STOCK MARKET IN CHINA

3.1 Introduction

Since the seminal work of Hamilton (1983), there is an expanding literature on the relationships between oil prices and economic activity. One stream of the literature focuses on the effects of oil prices on stock markets. In theory, an increase in oil price adversely impacts the profits of industries due to the increase in the production cost and thus has a negative effect on the stock market. But there is not a consistent conclusion in empirical findings. Park and Ratti (2008) estimate the effects of oil price shocks and oil price volatility on the real stock returns of the United States and 13 European countries using a multivariate VAR analysis. They find that oil price shocks have a statistically significant impact on real stock returns in the same month or within one month. Kilian and Park (2009) set up a model with four variables -- the percentage change in world crude oil production, global real economic activity, the real oil price, and return on US stocks -- and find that while oil demand shocks do depress stock prices, oil supply shocks have much less impact on stock prices. Apergis and Miller (2009) use a Structural VAR approach to analyze the effect of structural oil market shocks on the stock prices in eight developed economies. It is shown that oil market shocks do not have a very large or significant impact on the stock prices in these countries.

The relationship between oil market and currency markets has also received great attention. The link was noted as early as in Golub (1983) and Krugman (1983): an oil-importing country may experience exchange rate depreciation when oil prices rise, and appreciation when oil prices fall. Likewise, the potential impact of exchange rates on oil price movements, highlighted by

Bloomberg and Harris (1995), is based on the law of one price for tradable goods: since oil is a homogeneous and internationally traded commodity, a change in the exchange rate would change the oil price to foreigners in foreign currencies, thereby changing their purchasing power and oil demand and, in turn, affecting the crude oil price. Amano and van Norden (1998) find a stable linkage exists between oil price shocks and the U.S. real effective exchange rate over the longer horizon. Their findings indicate that oil prices have been the dominant source of persistent shocks on real exchange rate. Chen and Chen (2007) investigate the long-run relationship between real oil prices and real exchange rates by using a monthly panel of G7 countries. They show that real oil prices may have been the dominant source of real exchange rate movements and that there is a cointegrating relationship between real oil prices and real exchange rates. Other studies confirming the significant impacts on real exchange rates in developed countries from oil price shocks include Chen and Rogoff (2003), Lizardo and Mollick (2010) and Zhou (1995).

Literature also suggests that a relationship between the stock market and the currency market may exist. For example, Dornbusch and Fischer (1980) show that as many companies borrow in foreign currencies to fund their operations, fluctuations in exchange rate affect the value of the earnings as well as the cost of its funds, and hence its stock price. Oskooee and Sohrabian (1992) test for the relationship between the S&P price index and the effective exchange rate of the dollar, finding bidirectional causality relationship between the two markets in the short-run, but not long-run cointegrating relationship. Ratner (1993) finds that the U.S. dollar exchange rate and U.S. stock prices are not related in the long-run. Abdala and Murinde (1997) examine exchange rate and stock prices interactions in emerging financial markets and show that the long-run relationship found only in India and Philippines while in the short-run, they found unidirectional causality from the exchange rates to stock prices in most of their sample countries. Doong, Yang, and Wang (2005) examine the dynamic relationship between stock prices and exchange rate in Asian countries, and find that stock prices and exchange rates are not cointegrated; they detect bidirectional causality in all sample countries except for Thailand.

Most of these studies focus on developed countries and few studies have been conducted in developing countries, especially in China. China became the world's second largest oil consumer in 2003, and in 2013, it replaced the United States as the world's largest net oil importer. During the past decade, the international price of crude oil has traveled from $50 per barrel in 2005 to a peak of $146 in 2009 and subsequently descended again to below $50 in 2015. It's very interesting to know how the changes in crude oil prices have affected the economy in China, and in particular whether the change in China's exchange rate policy has impacted this dynamic. Current studies include but are not limited to: Cong et al. (2008) investigate the interactive relationships between oil price shocks and Chinese stock market using multivariate vector auto-regression. They consider different stock indices and both world oil price shocks and China oil price shocks. They find that oil price shocks do not show statistically significant impact on the real stock returns of most Chinese stock market indices, except for manufacturing index and some oil companies. Both the world oil price shocks and China oil price shocks can explain much more than interest rates for manufacturing index. Li et al. (2012) investigate the relationship between oil prices and the Chinese stock market at the sector level. They confirm a panel cointegration relationship between oil prices and stock prices and find that the real oil price has a positive effect on stock market in the long run. Huang and Guo (2007) construct a four-dimensional structural VAR model, and find that real oil price shocks would lead to a minor appreciation of the long-term real exchange rate.

The goal of this paper is to fill an important gap in the literature by empirically investigating short-run and long-run relationship in a system of international crude oil price, Chinese stock market, currency market and other economic activities, but from a new angle -- we mainly want to compare different dynamics before and after the exchange rate regime shift in 2005. China's foray into the market economy started after the initiation of the reform and opening up policy in 1978. It is at this time that China's GDP started to see growth rates above 8% per year[1]. The de-regulation since late 1970s has been gradual enough that the Chinese economy is still controlled by the government to a large degree. In particular China's central bank is not independent of the government, unlike the Federal Reserve in the U.S.[2] As a result the China's central bank is charged primarily with preventing appreciation of the Chinese yuan (China's official currency) while avoiding severe effects on the rest of the economy. Up until the summer of 2005, the exchange rate of the yuan was measured purely against the dollar. After that time, the yuan has been compared to an undisclosed index of currencies, which has allowed the yuan to appreciate against the U.S. dollar. In this paper we choose to focus on the mid 1990's through 2014 and want to obtain a detailed picture of the dynamics of the economy to learn in detail how these dynamics are affected by the policy change in 2005.

The rest of this paper is organized as follows. Section 3.2 describes the choices of data. Section 3.3 presents the empirical results. Finally, section 3.4 concludes.

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[1] See Zhu (2012) for a detailed analysis.

[2] See Moskow and Lemieux (2008) for an excellent discussion.

3.2 The data

3.2.1 Choices of variables and time period

Broadly speaking, the initial goal of this paper was to analyze the changes brought to the Chinese economy by the change in exchange rate policy that occurred in 2005. This leads to two important questions: which variables should be used to describe the economy and what is the relevant time period.

When dealing with developed economies, the standard approach to the second question is to utilize all available data. With a rapidly changing country like China, this approach is problematic, as major reforms to the political and economic system are frequent. We are choosing to start looking at data from January 1996 onwards. The reason for the choice is that the late 1980s and early 1990s saw major liberalizations in China, from the lifting of price controls, trade restrictions and other regulations, to large scale privatization of state industries. The precise cut-off date that should be used is not obvious, as we are hoping to avoid transition effects while still having a reasonable amount of data available. Eventually we chose January 1996 as our starting date because by then inflation and hence real interest rates seem to have stabilized. For the end of the example we chose December 2014. We had the option of including one additional year of data, but the Chinese stock indices started rising rapidly at the beginning of 2014, and by early 2015 concerns about a crash were rising. This led to the Chinese government implementing many brand new interventions in the stock market in the hope of softening the landing (see Figure 3.1). As a result, we are yet again faced with a changing paradigm of policy and therefore choose to end the data in December of 2014.

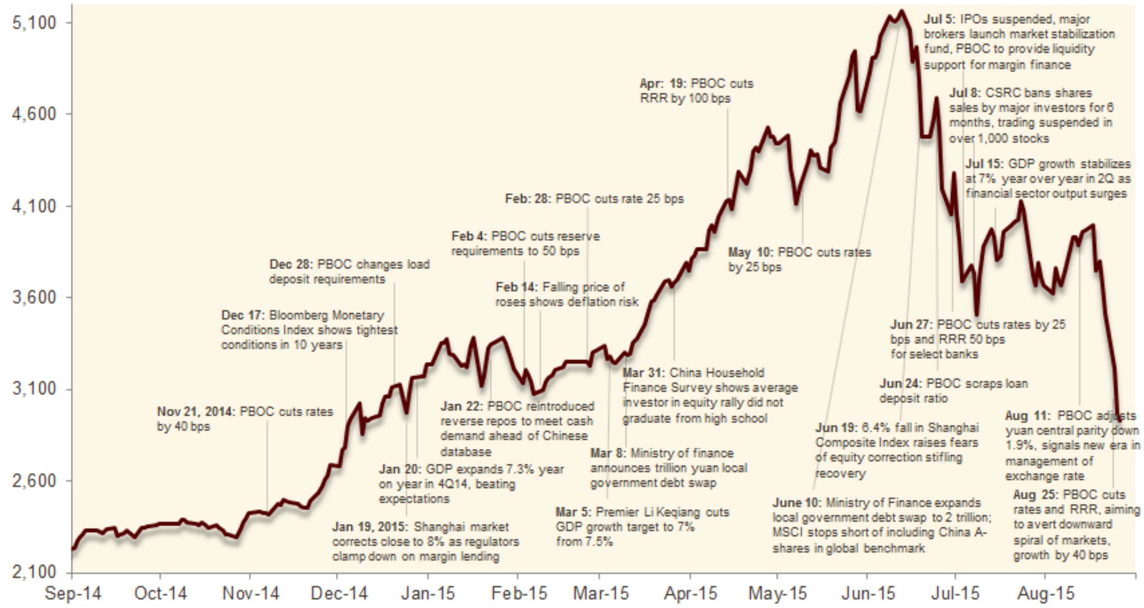

Figure 3.1. *China stock market and policies from Nov 2014 to Aug 2015*

The next choice to be made was which variables to include in the analysis. Multiple considerations are at play here. A primary restriction is that we cannot hope to identify a model with a significant number of endogenous variables. Pulling in the opposite direction is that fact that most economic variables will be in some way connected to exchange rate. We attempted to strike a balance on this issue, while being well aware that no two economists would likely come up with the same set of variables. The first variable we included is the oil price measured in yuan. Since the exchange rate used to be pinned to the US dollar prior to 2005, any price changes in oil were carried directly in the Chinese economy. After the exchange rate has been allowed to vary, it is likely that price changes in oil no longer are having a one for one impact. Therefore, the importance of the oil price as a direct determinant for various aspects of the Chinese macro economy is like to have been reduced. Secondly we wish to incorporate a measure of overall activity in the Chinese economy. While GDP is frequently used in this capacity, we decided to use industrial production.[3] There were several reasons for this choice. We would expect industrial production to be directly impacted by changes in oil price, but also by changes in the exchange rate because China is so closely linked to the rest of the world in terms of trade. The GDP is likely to incorporate many other relations less directly linked to the international prices. In addition, GDP is available only quarterly, while industrial production is available on a monthly basis. We also include the stock price because there is evidence in the literature (see section 3.1) that it is closely linked to oil prices and furthermore it seems natural that as industry has been privatized the stock market index should be linked to industrial production. Finally, we include the interest rate, as it is an indicator for alternative means of investment as well as the cost of borrowing for industry, and finally we include the exchange rate.

3.2.2 Description of the data

Following the literature, we are using real data as opposed to nominal. This choice implies that the exchange rate before 2005 is not fixed, but it is essentially a measure of inflation. The Chinese stock market is indicated by the monthly average value of Shanghai Stock Exchange (SSE) Composite Index, divided by the monthly CPI of China. The industrial production data is taken from the national Bureau of Statistics of China, measured as real growth rate (same month last year = 100).[4] The exchange rate is the growth rate of real exchange rate between US dollar

[3] Thank Lutz Kilian for suggestion.

[4] We wanted to use the industrial production level to measure the performance of Chinese economic activity. However, after 2006 the level data are no longer released by China's National Bureau of Statistics. And since 2011, enterprises above designated size was changed from 5 million (revenue from principal business) to 20 million, which makes the level data non-comparable. Also, the data on growth rate are released as year-on-year percent change. The underlying index series is not released. Therefore, we cannot calculate the month-to-month changes. Thank Min Wang and Xin Li for suggestion.

and Chinese yuan. For the crude oil prices, we first take the data of Europe Brent spot price FOB measured in dollars per barrel from the website of Energy Information Administration (EIA) (http://www.eia.gov), and then use the real exchange rate to adjust real (dollar) price of oil to obtain real oil price in yuan.[5] For the interest rate, we use the one-year loan real interest rate. All variables except for the industry production growth rate are seasonally adjusted.[6] Following the literature, the oil price and the stock index are analyzed with a logarithmic transformation. Graphical representations of the data series are presented in Figure 3.2 and Figure 3.3.

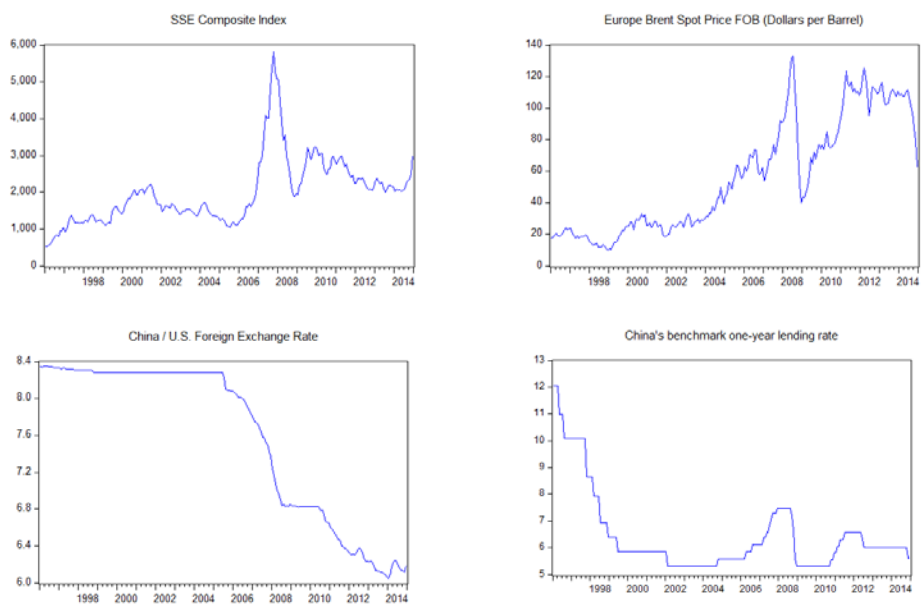

Figure 3.2. *Graphical representation of the nominal data*

[5] Since 2011, there has been a widening of the spread between the WTI and Brent prices, with WTI oil trading at a discount, reflecting a local glut of light sweet crude oil in the central United States driven by increased U.S. shale oil production. As a result, the WTI price of crude oil is no longer a representative for the price of oil in global markets, and it has become common to use the price of Brent crude oil as a proxy for the world price in recent years. (see Kilian 2016)

[6] The industrial production growth rate is measured as a year-on-year percent change, so we don't need to do seasonal adjustment on it.

Figure 3.3. *Graphical representation of the real data*

3.3 Empirical Results

3.3.1 Initial data analysis

To compare different dynamics before and after the exchange rate regime shift in July 2005, we divide the whole sample into two segments, one from January 1996 to July 2005, and the other from August 2005 to December 2014. Before proceeding to VAR modelling and impulse response analysis, we take a closer look at the individual series. In this analysis, we attempt to get a handle on trends, stationarity and breaks. This is made harder by the fact that there are no methods available which handle all three issues together, even though there is an abundance of methods looking at these issues in a pairwise manner. As a result, we choose to only incorporate those breaks which are clearly present in the data and backed up by knowledge of economic events causing them. Taking these as given we will proceed to examine the trend structure of the data series.

An initial visual examination suggests that there is a break in 2001 for the stock index and a break around 2008 for crude oil price and stock index. The one in 2001 was consistent with the fact that at that time the Chinese government issued rules requiring listed firms to sell some state shares in IPOs and additional share offers, and give the money to the national pension fund, which sparked a four-year stock market slump. The break in 2008 is consistent with the Great Recession. We take these as given as we proceed.

To identify the trend structure, we apply the trend test of Perron and Yabu (2009a) which is valid regardless of whether or not the series is stationary. The method is described in detail in the appendix. The results are reported in Table 3.1. The trend tests suggest that there is no time trend for any of the variables in each subsample, and we can now proceed to test for unit roots. [7]

Table 3.2 presents the results for unit root tests. For stock index in the first subsample, we include a break in June 2001. For stock index and oil price in the second subsample, we include a break in December 2008. For these three series, we apply the unit root test with breaks which follows the basic framework outlined in Perron (1989). The critical values are taken from Perron

[7] If we take into account of the breaks in 2001 and 2008, the "no trend" results still hold for stock and oil in the second subsample. But the test suggests a time trend exists and changes at 2001 for stock in the first subsample. However, this is not going to change the results of unit root tests.

Table 3.1. Trend test

1996M1-2005M7

	ln_stock	industry_growth	ln_oil	exchange_growth	interest
slope	0.0048	0.0342	0.0085	-0.0011	-0.0115
t-stat	0.6424	0.3617	1.0656	-0.2302	-0.1853

2005M8-2014M12

	ln_stock	industry_growth	ln_oil	exchange_growth	interest
slope	0.0061	-0.0723	-0.0040	0.0137	-0.0080
t-stat	0.5325	-0.5709	-0.3869	1.0364	-0.0814

and Vogelsang (1993)[8]. For other series, the standard Augmented Dickey-Fuller unit root test is applied. The critical values are taken from MacKinnon (1996). In both cases, the modified Akaike criterion (MAIC) is used to select the lag length.

The unit root tests suggest all three series are $I(1)$. We are now ready to proceed to multivariate modelling of the system.

3.3.2 VAR model

Initially we estimate a reduced for VAR model in levels for the two separate time periods. Here we do not include the breaks as we are hoping to obtain a model which to some degree explains and maps out the consequence of these economic events. We use Maximum Likelihood to estimate the VAR and then proceed to test for the number of cointegration relationships using the Johansen's (1988, 1996) trace test. When we fit the VAR model in the levels of the data, we

[8] There was an error in one of the key tables of critical values in Perron (1989), and this was later corrected by Perron and Vogelsang (1993).

Table 3.2. Unit root test

Level

	Series	Time trend	Breaks	t-Statistic	0.01	0.05	0.1
1996M1-2005M7	ln_stock	no	2001/06	-2.30	4.43	3.76	3.47
	industry_growth	no	no	-1.65	3.49	2.89	2.58
	ln_oil	no	no	-0.47	3.49	2.89	2.58
	exchange_growth	no	no	-2.47	3.49	2.89	2.58
	interest	no	no	-1.24	3.49	2.89	2.58
2005M8-2014M12	ln_stock	no	2008/12	-2.21	4.35	3.73	3.45
	industry_growth	no	no	-1.83	3.49	2.89	2.58
	ln_oil	no	2008/12	-1.85	4.35	3.73	3.45
	exchange_growth	no	no	-2.00	3.49	2.89	2.58
	interest	no	no	-2.18	3.49	2.89	2.58

First difference

	Series	Time trend	Breaks	t-Statistic	0.01	0.05	0.1
1996M1-2005M7	D(ln_stock)	no	no	7.85***	2.59	1.94	1.61
	D(industry_growth)	no	no	10.57***	2.59	1.94	1.61
	D(ln_oil)	no	no	-1.82*	2.59	1.94	1.61
	D(exchange_growth)	no	no	15.80***	2.59	1.94	1.61
	D(interest)	no	no	16.56***	2.59	1.94	1.61
2005M8-2014M12	D(ln_stock)	no	no	-2.50**	2.59	1.94	1.61
	D(industry_growth)	no	no	9.21***	2.59	1.94	1.61
	D(ln_oil)	no	no	4.92***	2.59	1.94	1.61
	D(exchange_growth)	no	no	16.28***	2.59	1.94	1.61
	D(interest)	no	no	22.18***	2.59	1.94	1.61

Note: *,** and *** denote statistical significance at the 10%, 5% and 1% level.

use the usual information criteria to determine the optimal maximum lag length. We also test for serial independence of the errors using LM tests. The requirement for serial independence suggests setting $p=5$ for the first subsample and $p=3$ for the second. We report the results in Table 3.3.

Table 3.3 Cointegration test

1996M1-2005M7

Hypothesis	Eigenvalue	LR	0.05 Critical Value	Prob.
r=0	0.30	97.08	76.97	0.00
r≤1	0.27	58.32	54.08	0.02
r≤2	0.12	23.16	35.19	0.52
r≤3	0.06	8.63	20.26	0.77
r≤4	0.02	2.31	9.16	0.72

Trace test indicates 2 cointegrating eqn(s) at the 5% level.

2005M8-2014M12

Hypothesis	Eigenvalue	LR	0.05 Critical Value	Prob.
r=0	0.36	109.96	76.97	0.00
r≤1	0.23	60.71	54.08	0.01
r≤2	0.17	31.92	35.19	0.11
r≤3	0.06	11.34	20.26	0.51
r≤4	0.04	4.32	9.16	0.37

Trace test indicates 2 cointegrating eqn(s) at the 5% level.

In each segment, it shows the existence of two cointegrating vectors with the trace test. At this high level at least, there does not seem to be a change in the model due to the 2005 policy change. Given the short time span of our data (approximately 9 years) we do not explore the long run relationship in any greater detail and proceed to examine short run dynamics.

3.3.3 Impulse response functions

Besides the long run equilibrium relationship, we are also interested in the short run dynamics between these variables. More specifically, we want to use the impulse response function to capture the effects of a positive shock from one variable to other variables. In this section, we use two methods to calculate the impulse response functions. The first one is the traditional method, which includes estimating a structural VAR model and then finding the impulse responses from that model. The second method is the local-projection estimation in Jorda (2005), which doesn't require any specifications on the model.

3.3.3.1 Identification of the structural VAR model

The reduced form VAR is given by

$$y_t = c + \sum_{i=1}^{p} A_i y_{t-i} + e_t,$$

where $y_t = (\text{ln\_stock, industry\_growth, ln\_oil, exchange\_growth, interest})'$, c is a vector of constants, p is the number of lags, $\{A_i\}_{i=1}^{p}$ are the 5×5 parameter coefficient matrices, and e_t is a vector of error terms. Let $\Sigma = E(e_t e_t^{'})$ be the residual covariance matrix. Following Amisano and Giannini (1997), the errors of the structural SVAR model can be written as:

$$Ae_t = Bu_t$$

where u_t is a vector of length $k = 5$, and represents the unobserved structural innovations. u_t is assumed to be orthonormal, that is, its covariance matrix is an identity matrix, $E(u_t u_t^{'}) = I$. A and B are 5×5 matrices to be estimated. Here in our model, we only add short-run restrictions on contemporaneous relationship but no long-run restrictions.[9] Based on economic intuition, we impose the following restrictions to obtain identification:

[9] Christiano et al. (2007) point out that structural VARs based on short-run restrictions perform well.

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We assume B to be an identity matrix, which means the SVAR considered has instantaneously uncorrelated residuals, that is, the structural shocks hitting the system are assumed to be mutually uncorrelated. The restrictions on A are motivated as follows.

(1) We allow stock index to contemporaneously react to shocks of oil price, exchange rate and interest rate, but not the industrial production in the same month. This is because financial markets usually react quickly to the shocks in other markets and there is no delay for the adjustment. For example, a decrease in the real interest rate will encourage people to invest in the stock market and make the stock index increase. An appreciation of Chinese yuan along with speculative capital inflows will also flourish the stock market. And a shock from oil price will affect the price of energy based equities. It's unlikely that the stock price reacts immediately within the same month with a change to the industrial production growth rate because usually the performance in the industrial production sectors cannot be accessed by the public until the government releases the report. And this argument applies to other equations. Of course, lagged values of industrial production growth rate are allowed to have effects on other variables in the system after the government releases the reports.

(2) The industrial production growth rate is allowed to be affected contemporaneously by oil price, exchange rate and interest rate. Crude oil is the most important energy sources used in production, and the interest rate affects the amount of capital that can be used in the production.

(3) The equation for oil price takes international crude oil price as being contemporaneously exogenous to the other variables in the system except for the exchange rate. We assume that oil price in dollar does not contemporaneously react to shocks to other variables in the system within a month -- it is determined by the global supply and demand of crude oil. In our model, since we measure the oil price in Chinese yuan, the price will definitely react immediately within the same month to a change to the real exchange rate.

(4) Changes in oil prices and interest rate result in a contemporaneous change in the real exchange rate. A shock to the interest rate may cause the instantaneous inflows or outflows of the speculative money, and thus affect the exchange rate and the exchange rate may change to soften the impact of changing oil prices.

(5) In the last equation, the real interest rate is allowed to react contemporaneously to an exchange rate shock.

We calculate impulse response functions and confidence intervals using the vars package in R. The confidence intervals provided in the package are known to have poor coverage (see Pesavento and Rossi 2007) and they are wide enough that, if taken seriously, there are hardly any significant impulse responses. It is not clear, however that a more suitable method exists for a structural VAR where the variables might be unit root processes. The most promising might be Pesavento and Rossi (2006) but these are only valid for long horizons, in our case 10 or more periods ahead, which is not useful for this application. Another problem with this standard two-step procedure is that it is justified only if the model coincides with the DGP, which is usually unknown.

3.3.3.2 Local-projection IRFs

Jorda (2005) introduces a method to compute impulse responses without specification and estimation of the underlying multivariate dynamic system. The central idea is estimating local projections at each period of interest. A brief description of their model is in the following.

The impulse responses are defined as the difference between two forecasts:

$$IR(t,s,d_i) = E(y_{t+s} \mid v_t = d_i; X_t) - E(y_{t+s} \mid v_t = 0; X_t), s = 0,1,2,... \qquad (3.1)$$

where y_t is an $n \times 1$ vector ($n = 5$ in our model); $X_t = (y_{t-1}, y_{t-2},...)'$, v_t is the $n \times 1$ vector of reduced-form disturbances; and D is an $n \times n$ matrix, whose columns d_i contain the relevant experimental shocks.

Run the regression[10]

$$y_{t+s} = \alpha^s + B_1^{s+1} y_{t-1} + B_2^{s+1} y_{t-2} + ... + B_p^{s+1} y_{t-p} + u_{t+s}^s, s = 0,1,2,...,h \qquad (3.2)$$

where α^s is an $n \times 1$ vector of constants, and B_i^{s+1} are matrices of coefficients for each lag i and horizon $s+1$. The collection of h regressions in (3.2) is denoted as local projections. Then the impulse responses from the local-linear projections in (3.2) are

$$\widehat{IR}(t,s,d_i) = \widehat{B_1^s} d_i, s = 0,1,2,...,h$$

with the normalization $B_1^0 = I$.

Impulse responses can be calculated by univariate regression methods with a heteroscedasticity and autocorrelation (HAC) robust estimator, with little loss of efficiency. Valid

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[10] Jorda (2005) points out that consistency does not require that the sequence of h system regressions in (3.2) be estimated jointly -- the impulse response for the j th variable in y_t can be estimated by a univariate regression of y_{jt} onto X_t. He also points out that the maximum lag p (which is determined by information criteria) need not be common to each horizon s .

inference for local projection impulse responses can be obtained with HAC robust standard errors. For example, let $\widehat{\Sigma}_L$ be the estimated HAC, variance-covariance matrix of the coefficients $\widehat{B_1^s}$ in expression (3.2); then a 95% confidence interval for each element of the impulse response at time can be constructed approximately as $1.96\times\left(d_i'\widehat{\Sigma}_L d_i\right)$.

3.3.3.3 Discussions

We find that the standard IRFs indicate statistically insignificant effects in most of the cases. [11] In this section, we mainly rely on the results by local-projection methods and focus on the effects from shocks on stock market, oil price and currency market, which is the main interest of the paper.

A positive stock price shock causes a statistically significant increase on itself, and the shocks dissipate more rapidly in the sample after 2005 (Figure 3.4). We also find a positive effect from stock market to the exchange rate for both subsamples. The effects before 2005 are moderate and it takes about 8 months to become insignificant. In contrast, after 2005 the effects are stronger but quickly dissipate (Figure 3.5). They all indicate that the financial market became more efficient with a flexible exchange rate regime. A positive shock in oil price has short significant effects on currency market and interest rate both before and after 2005 (Figure 3.6 and Figure 3.7). The effects are stronger after 2005. Specifically, if the oil price increases, it causes a depreciation of yuan, and a decrease in interest rate. This is because when oil price increases, the demand for

[11] Figures of all IRFs for both cases are available from the author upon request. In this paper we only report figures which are discussed in this section.

foreign currency increases, which makes the exchange rate increases.[12] Also, the increasing cost depresses the production and makes the interest rate decrease. A positive shock in exchange rate (i.e., a depreciation in yuan) has a short significant effect on the interest rate and the effect is stronger before 2005 (Figure 3.8). A positive shock in interest rate has a short negative significant effect on exchange rate (yuan appreciates) (Figure 3.9). This is because when interest rate increases, the demand for yuan increases.

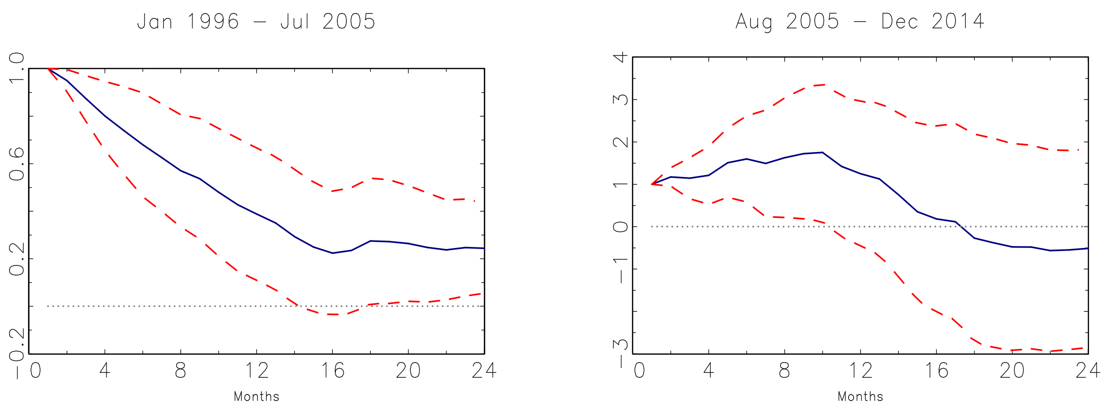

Figure 3.4. *IRF of shocks from stock index to stock index*

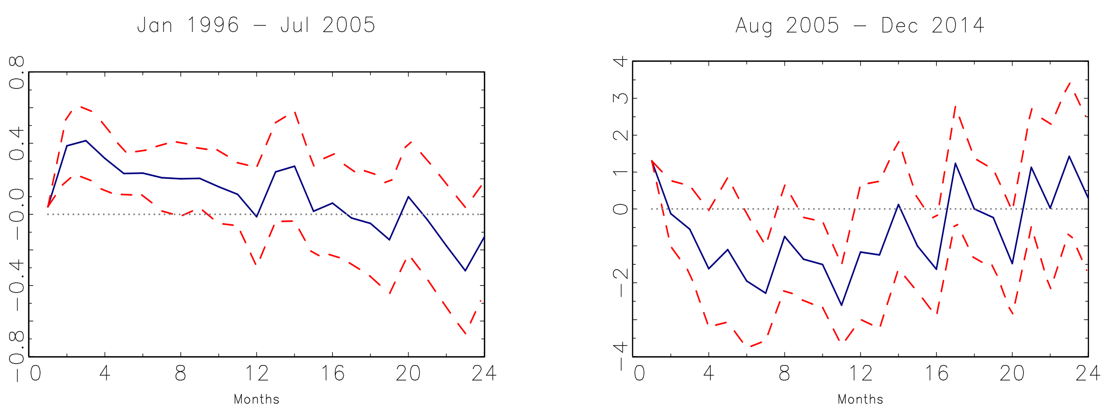

Figure 3.5. *IRF of shocks from stock index to growth rate of exchange rate*

[12] The exchange rate is measured as how much yuan can be traded for 1 US dollars, so the increase in exchange rate means yuan depreciation.

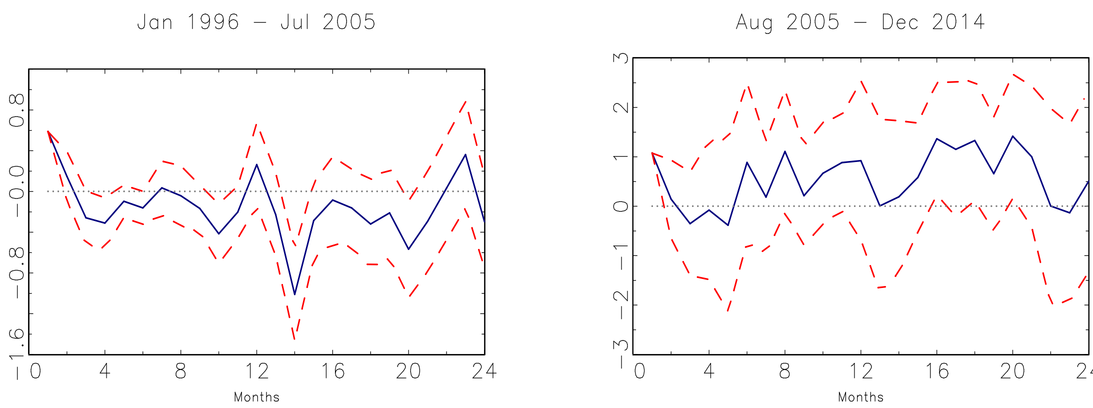

Figure 3.6. *IRF of shocks from oil price to growth rate of exchange rate*

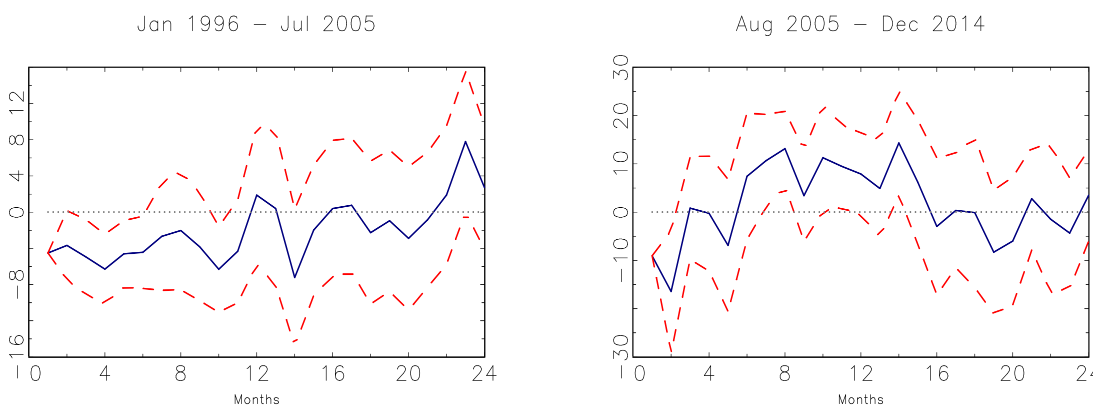

Figure 3.7. *IRF of shocks from oil price to interest rate*

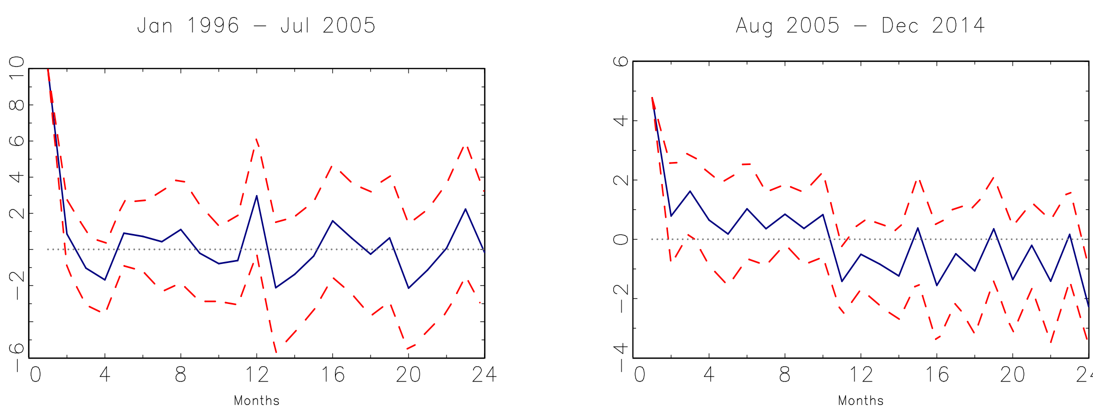

Figure 3.8. *IRF of shocks from growth rate of exchange rate to interest rate*

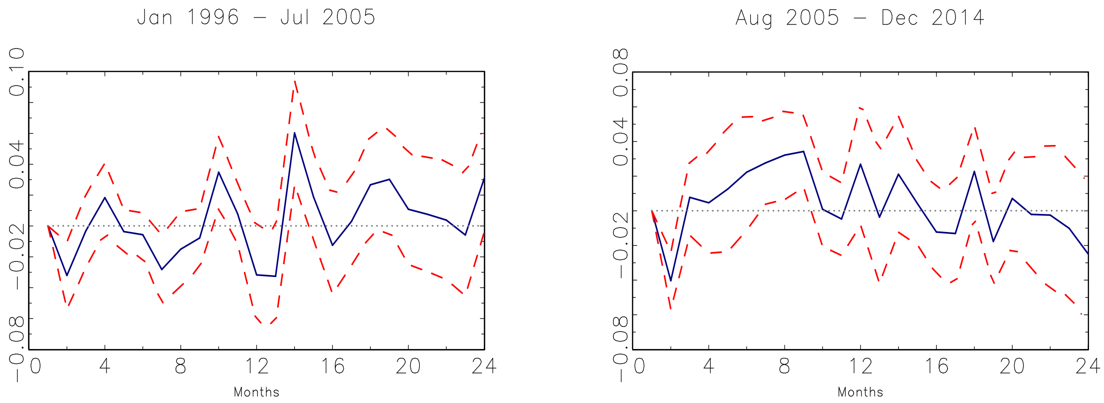

Figure 3.9. *IRF of shocks from interest rate to growth rate of exchange rate*

3.4 Conclusions

This paper investigates the long-run cointegrating relationship and short-run dynamics in a system of crude oil prices, Shanghai Stock Exchange Composite Index, industrial production growth rate, real exchange rates and real interests rate for the period January 1996 to December 2014. To figure out the effects of exchange rate regime shift in July 2005, we divide the whole sample to two periods, one from January 1996 to July 2005, and the other from August 2005 to December 2014. In each subsample, we use the newly developed Perron-Yabu trend test to examine the time trend properties without knowing whether the noise component is $I(0)$ or $I(1)$. Johansen cointegration tests confirm two cointegrating equations among these five variables in each segment. Finally, we construct structural VAR models and compare impulse response functions in each subsample.

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APPENDIX

TREND TEST

To test for long-run cointegrating relationship, we first need to test for unit root, and it's important to know whether there is a time trend in each series. There is a large literature on trend test, most of which requires a stationary process. But in most of the cases, the noise component can be either $I(0)$ or $I(1)$ and that in general no a priori knowledge about this is available. Perron and Yabu (2009a) propose a test for the slope of a trend function when it is a priori unknown whether the series is trend-stationary or contains an autoregressive unit root. A brief description of their model is in the following.

The data-generating process for a scalar random variable y_t is assumed to be

$$\begin{aligned} y_t &= x_t'\Psi + u_t, \\ u_t &= \alpha u_{t-1} + e_t. \end{aligned}$$

for $t = 1,...,T$, where $e_t \sim iid\left(0,\sigma^2\right)$, $x_t = (1,t)'$are deterministic components, and $\Psi = (\mu, \beta)'$ is unknown. Here $-1 < \alpha \le 1$, so that both stationary and integrated errors are allowed. The null hypothesis is $\beta = 0$. The procedure is the following:[13]

1. Detrend the data by OLS on $y_t = x_t'\Psi + u_t$ to obtain the residuals $\hat{u}_t$.

2. Use $\hat{u}_t$ to construct the weighted symmetric least-square (WSLS) estimate

$$\widehat{\alpha}_W = \frac{\sum_{t=2}^{T} \hat{u}_t \hat{u}_{t-1}}{\sum_{t=2}^{T-1} \hat{u}_t^2 + T^{-1} \sum_{t=1}^{T} \hat{u}_t^2}.$$

An estimate of its variance is given by

[13] See Perron and Yabu (2009a) Page 59.

$$\hat{\sigma}_W = \frac{\sum_{t=2}^{T}\left(\hat{u}_t - \hat{\alpha}_W \hat{u}_{t-1}\right)^2}{(T-3)\left(\sum_{t=2}^{T-1}\hat{u}_t^2 + T^{-1}\sum_{t=1}^{T}\hat{u}_t^2\right)}$$

and the associated t-ratio for testing that $\alpha = 1$ is $\hat{\tau}_W = \left(\hat{\alpha}_W - 1\right)/\hat{\sigma}_W$.

3. Get the truncated estimate $\hat{\alpha}_{MU} = \hat{\alpha}_W + C\left(\hat{\tau}_W\right)\hat{\sigma}_W$, where

$$C\left(\hat{\tau}_W\right) = \begin{cases} -\hat{\tau}_W & \text{if } \hat{\tau}_W > \tau_{pct} \\ T^{-1}\hat{\tau}_W - (1+r)\left[\hat{\tau}_W + K\left(\hat{\tau}_W + a\right)\right]^{-1} & \text{if } -a < \hat{\tau}_W \le \tau_{pct} \\ T^{-1}\hat{\tau}_W - (1+r)\left(\hat{\tau}_W\right)^{-1} & \text{if } -\left[(1+r)T\right]^{1/2} < \hat{\tau}_W \le -a \\ 0 & \text{if } \hat{\tau}_W \le -\left[(1+r)T\right]^{1/2} \end{cases}$$

with r the number of parameters estimated in the trend function, i.e., the number of elements in the vector Ψ, and $K = \frac{(1+r)T - \tau_{pct}^2\left(I_p + T\right)}{\tau_{pct}\left(a + \tau_{pct}\right)\left(I_p + T\right)}$, where $I_p = 1$[14] and τ_{pct} is a percentile of the limiting distribution of $\hat{\tau}_W$ when $\alpha = 1$. Here we choose $a = 5$ and $\tau_{pct} = -1.96$, the median of the distribution of $\hat{\tau}_W$ when $\alpha = 1$, then $\hat{\alpha}_{MU}$ is approximately median unbiased, in the sense that it is nearly unbiased when $\alpha < 1$ and has a median of 1 when $\alpha = 1$.[15]

4. Apply the truncation

$$\hat{\alpha}_{MS} = \begin{cases} \hat{\alpha}_{MU}, & \text{if } \left|\hat{\alpha}_M - 1\right| > T^{-1/2} \\ 1, & \text{if } \left|\hat{\alpha}_M - 1\right| \le T^{-1/2} \end{cases}.$$

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[14] In the case of a general noise component with an $AR(p)$ structure, I_p is the integer part of $\frac{p+1}{2}$.

[15] Perron and Yabu (2009a) also consider using $\tau_{pct} = -2.85$ as suggested by Roy et al. (2004), which is approximately the 85th percentile of the distribution $\hat{\tau}_W$ when $\alpha = 1$. However, Perron and Yabu (2009a) also point out that $\tau_{pct} = -1.96$ is in general preferable (higher power and good size) unless one is worried about facing a process with a strong negative moving-average component. So in this study, we choose $\tau_{pct} = -1.96$.

5. Apply a GLS procedure with $\widehat{\alpha}_{MS}$ to obtain the estimates of the trend parameter β, that is, apply OLS to the regression

$$\left(1-\widehat{\alpha}_{MS}L\right)y_t = \left(1-\widehat{\alpha}_{MS}L\right)x_t\,'\Psi + e_t, \text{ for } t = 2,...,T$$
$$y_1 = x_t\,'\Psi + u_1$$

and construct the standard t-statistic which we shall demote by t_{β}^{FS} .